

LOG CANONICAL THRESHOLDS OF CERTAIN FANO HYPERSURFACES

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ABSTRACT. We study log canonical thresholds on quartic threefolds, quintic fourfolds, and double spaces. As an application, we show that they have a Kähler-Einstein metric if they are general.

1. INTRODUCTION.

All varieties are defined over \mathbb{C} .

1.1. Introduction. The multiplicity of a nonzero polynomial $f \in \mathbb{C}[z_1, \dots, z_n]$ at a point $P \in \mathbb{C}^n$ is the nonnegative integer m such that $f \in \mathfrak{m}_P^m \setminus \mathfrak{m}_P^{m+1}$, where \mathfrak{m}_P is the maximal ideal of polynomials vanishing at the point P in $\mathbb{C}[z_1, \dots, z_n]$. It can be also defined by derivatives. The multiplicity of f at the point P is the number

$$\text{mult}_P(f) = \min \left\{ m \mid \frac{\partial^m f}{\partial^{m_1} z_1 \partial^{m_2} z_2 \dots \partial^{m_n} z_n}(P) \neq 0 \right\}.$$

On the other hand, we have a similar invariant that is defined by integrations. This invariant, which is called the log canonical threshold of f at the point P , is given by

$$c_P(f) = \sup \left\{ c \mid |f|^{-c} \text{ is locally } L^2 \text{ near the point } P \in \mathbb{C}^n \right\}.$$

This number appears in many places. For instance, the log canonical threshold of the polynomial f at the origin is the same as the absolute value of the largest root of the Bernstein-Sato polynomial of f .

Even though log canonical threshold was implicitly known, it was formally introduced by V. Shokurov in [31] as follows. Let X be a \mathbb{Q} -factorial variety with at worst log canonical singularities, $Z \subset X$ a closed subvariety, and D an effective \mathbb{Q} -divisor on X . The log canonical threshold of D along Z is the number

$$c_Z(X, D) = \sup \left\{ c \mid \text{the log pair } (X, cD) \text{ is log canonical along } Z \right\}.$$

For the case $Z = X$ we use the notation $c(X, D)$ instead of $c_X(X, D)$. Because log canonicity is a local property, we see that

$$c_Z(X, D) = \inf_{P \in Z} \{c_P(X, D)\}.$$

If $X = \mathbb{C}^n$ and $D = (f = 0)$, then we also use the notation $c_0(f)$ for the log canonical threshold of D near the origin.

Even though several methods have been invented in order to compute log canonical thresholds, it is not easy to compute them in general. However, many problems in birational geometry are related to log canonical thresholds. The log canonical thresholds play a significant role in the study on birational geometry. They show many interesting properties (see [10], [11], [13], [19], [20], [21], [24], [25], [26], [27], [22]).

We occasionally find it useful to consider the smallest value of log canonical thresholds of effective divisors linearly equivalent to a given divisor, in particular, an anticanonical divisor (for instance, see [24]).

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Definition 1.1.1. Let X be a \mathbb{Q} -factorial Fano variety with at worst log terminal singularities. For a natural number $m > 0$, we define the m -th global log canonical threshold of X by the number¹

$$\mathrm{lct}_m(X) = \inf \left\{ c \left(X, \frac{1}{m} H \right) \mid H \in |-mK_X| \right\}.$$

Also, we define the global log canonical threshold of D by the number

$$\mathrm{lct}(X) = \inf_{n \in \mathbb{N}} \left\{ \mathrm{lct}_n(X) \right\}.$$

We can immediately see

$$\mathrm{lct}(X) = \sup \left\{ c \mid \begin{array}{l} \text{for every } \mathbb{Q}\text{-divisor } D \text{ with } D \equiv -K_X \\ \text{the log pair } (X, cD) \text{ is log canonical} \end{array} \right\}.$$

To see the simplest case, let S be a smooth del Pezzo surface. It follows from [4] and [24] that²

$$(1.1.2) \quad \mathrm{lct}(S) = \mathrm{lct}_1(S) = \begin{cases} 1/3 & \text{when } S \cong \mathbb{F}_1 \text{ or } K_S^2 \in \{7, 9\}, \\ 1/2 & \text{when } S \cong \mathbb{P}^1 \times \mathbb{P}^1 \text{ or } K_S^2 \in \{5, 6\}, \\ 2/3 & \text{when } K_S^2 = 4, \\ 2/3 & \text{when } S \text{ is a cubic in } \mathbb{P}^3 \text{ with an Eckardt point,} \\ 3/4 & \text{when } S \text{ is a cubic in } \mathbb{P}^3 \text{ without Eckardt points,} \\ 3/4 & \text{when } K_S^2 = 2 \text{ and } |-K_S| \text{ has a tacnodal curve,} \\ 5/6 & \text{when } K_S^2 = 2 \text{ and } |-K_S| \text{ has no tacnodal curves,} \\ 5/6 & \text{when } K_S^2 = 1 \text{ and } |-K_S| \text{ has a cuspidal curve,} \\ 1 & \text{when } K_S^2 = 1 \text{ and } |-K_S| \text{ has no cuspidal curves.} \end{cases}$$

For a quasismooth hypersurface X in $\mathbb{P}(a_0, \dots, a_4)$ of degree $\sum_{i=0}^4 a_i - 1$, where $a_0 \leq \dots \leq a_4$, one can find $\mathrm{lct}(X) > \frac{3}{4}$ for 1936 values of $(a_0, a_1, a_2, a_3, a_4)$ (see [18]). Moreover, for a quasismooth hypersurface X in $\mathbb{P}(1, a_1, \dots, a_4)$ of degree $\sum_{i=1}^4 a_i$ having terminal singularities, there are exactly 95 possible quadruples (a_1, a_2, a_3, a_4) found in [16] and [18]. The paper [3] shows $\mathrm{lct}(X) = 1$ if

$$(a_1, a_2, a_3, a_4) \notin \left\{ (1, 1, 1, 1), (1, 1, 1, 2), (1, 1, 2, 2), (1, 1, 2, 3) \right\}$$

and the hypersurface X is sufficiently general.

The papers [14] and [15] show that the global log canonical threshold of a rational homogeneous space of Picard rank 1 and Fano index r is $\frac{1}{r}$.

Example 1.1.3. Let X be a smooth hypersurface of degree $n + 1 \geq 3$ in \mathbb{P}^{n+1} . Then

$$\mathrm{lct}_m(X) \geq \frac{n}{n+1}$$

due to [2] and [5]. Furthermore, $\mathrm{lct}_m(X) = \frac{n}{n+1}$ if and only if X contains a cone of dimension $n - 1$ (see [2], [5], and [11]). The inequality obviously implies that

$$\mathrm{lct}(X) \geq \frac{n}{n+1}.$$

However, the paper [30] shows that $\mathrm{lct}(X) = 1$ if X is general and $n \geq 5$.

From Example 1.1.3, it is natural to expect the following:

¹The number $\mathrm{lct}_m(X)$ is undefined if the linear system $|-mK_X|$ is empty.

²A point of a cubic surface is called an Eckardt point if the cubic contains 3 lines passing through this point. A general cubic surface has no Eckardt point.

Conjecture 1.1.4. *The global log canonical thresholds of a general quartic threefold and a general quintic fourfold are 1.*

This conjecture has been proposed for canonical thresholds in [30].

For an evidence of the conjecture, we can consider the first global log canonical threshold of a general hypersurface. It is not hard to show that the first global log canonical threshold of a general hypersurface of degree $n + 1 \geq 4$ in \mathbb{P}^{n+1} is one (see Proposition 2.1.1). In the case of smooth quartic threefolds, we can find all the first global log canonical thresholds.

Theorem 1.1.5. *Let X be a smooth quartic threefold in \mathbb{P}^4 . The first global log canonical threshold $\text{lct}_1(X)$ is one of the following:*

$$\left\{ \frac{3}{4}, \frac{29}{36}, \frac{22}{27}, \frac{5}{6}, \frac{16}{19}, \frac{17}{20}, \frac{6}{7}, \frac{13}{15}, \frac{37}{42}, \frac{7}{8}, \frac{8}{9}, \frac{9}{10}, \frac{23}{26}, \frac{11}{12}, \frac{12}{13}, \frac{13}{14}, \frac{14}{15}, \frac{15}{16}, \frac{31}{34}, \frac{17}{18}, \frac{21}{22}, \frac{23}{24}, \frac{29}{30}, \frac{41}{42}, 1 \right\}.$$

Furthermore, for each number μ in the set above, there is a smooth quartic threefold X with $\text{lct}_1(X) = \mu$.

For the global log canonical thresholds, we prove the following:

Theorem 1.1.6. *Let X be a general hypersurface of degree $n = 4$ or 5 in \mathbb{P}^n . Then*

$$\text{lct}(X) \geq \begin{cases} \frac{16}{21} & \text{for } n = 4; \\ \frac{22}{25} & \text{for } n = 5. \end{cases}$$

The global log canonical threshold of a Fano variety is an algebraic counterpart of the α -invariant introduced in [32]. One of the most interesting applications of global log canonical thresholds of Fano varieties is the following result proved in [9] (see also [23] and [32]).

Theorem 1.1.7. *Let X be an n -dimensional Fano variety with at most quotient singularities. The variety X has an orbifold Kähler–Einstein metric if the inequality*

$$\text{lct}(X) > \frac{n}{n+1}$$

holds.

Taking Theorem 1.1.7 into consideration, we see that (1.1.2) implies the existence of a Kähler–Einstein metric on a general cubic surface and that the paper [30] implies the existence of a Kähler–Einstein metric on a general hypersurface of degree $n + 1 \geq 6$ in \mathbb{P}^{n+1} . Even though Theorem 1.1.6 is much weaker than Conjecture 1.1.4, they are strong enough to imply the existence of a Kähler–Einstein metric. Consequently, we can obtain the following:

Corollary 1.1.8. *A general hypersurface of degree $n + 1 \geq 3$ in \mathbb{P}^{n+1} has a Kähler–Einstein metric.*

Also, in this paper, we will study log canonical thresholds on double spaces³ and obtain similar results as what we have on Fano hypersurfaces in \mathbb{P}^n . For instance, we will prove the following:

Theorem 1.1.9. *Let V be the smooth double cover of \mathbb{P}^3 ramified along a sextic. Then, the first global log canonical threshold of the Fano variety V is one of the following:*

$$\left\{ \frac{5}{6}, \frac{43}{50}, \frac{13}{15}, \frac{33}{38}, \frac{7}{8}, \frac{33}{38}, \frac{8}{9}, \frac{9}{10}, \frac{11}{12}, \frac{13}{14}, \frac{15}{16}, \frac{17}{18}, \frac{19}{20}, \frac{21}{22}, \frac{29}{30}, 1 \right\}.$$

Furthermore, for each number μ in the set above, there is a smooth double cover V of \mathbb{P}^3 ramified along a sextic with $\text{lct}_1(V) = \mu$.

³A double space is a double cover of \mathbb{P}^n ramified along a hypersurface of degree $2m$. It can be considered as a hypersurface of degree $2m$ in the weighted projective space $\mathbb{P}(1^{n+1}, m)$.

Furthermore, we will see that the first global log canonical threshold of a smooth double space is equal to its global log canonical threshold (see Proposition 3.3.1), and hence we can have the same statement for the global log canonical threshold of the Fano variety V as Theorem 1.1.9. (See Corollary 3.3.3).

Let us close this section by a conjecture inspired by Question 1 in [33].

Conjecture 1.1.10. *For a Fano variety X , $\text{lct}(X) = \text{lct}_m(X)$ for some natural number $m \geq 1$.*

1.2. Basic tools. Before we proceed, we review a few of useful facts for log canonical thresholds.

Lemma 1.2.1. *Let f be a holomorphic function near $0 \in \mathbb{C}^n$ and $D = (f = 0)$. Set $d = \text{mult}_0 f$ and let f_d denote the degree d homogeneous part of the Taylor series of f . Let $T_0 D = (f_d = 0) \subset \mathbb{C}^n$ be the tangent cone of D and $\mathbb{P}(T_0 D) = (f_d = 0) \subset \mathbb{P}^{n-1}$ the projectivised tangent cone of D . Then*

- (1) $\frac{1}{d} \leq c_0(f) \leq \frac{n}{d}$.
- (2) $c_0(f_d) \leq c_0(f)$.
- (3) *The log pair $(\mathbb{P}^{n-1}, \frac{n}{d}\mathbb{P}(T_0 D))$ is log canonical if and only if $c_0(f) = \frac{n}{d}$.*
- (4) *If $\mathbb{P}(T_0 D)$ is smooth (or even log canonical) then $c_0(f) = \min\{1, \frac{n}{d}\}$.*

Proof. See [19]. □

Lemma 1.2.2. *Let f be a holomorphic function near $0 \in \mathbb{C}^n$. Assign rational weights $w(x_i)$ to the variables and let $w(f)$ be the weighted multiplicity of f (=the lowest weight of the monomials occurring in f). Let f_w denotes the weighted homogeneous leading term of the Taylor series of f . Then,*

- (1) $c_0(f_w) \leq c_0(f)$.
- (2) $c_0(f) \leq \frac{\sum w(x_i)}{w(f)}$. *The equality holds if the log pair $(\mathbb{C}^n, q(f_w = 0))$ is log canonical in the outside of the origin, where $q = \frac{\sum w(x_i)}{w(f)}$.*

Proof. See [19] and [20]. □

For curve cases, the following description in terms of Newton polygon will be helpful for us to find appropriate weights and leading terms.

Lemma 1.2.3. *Fix a power series f in coordinates x and y . Then as $w(x)$ and $w(y)$ ranges over all possible choices for weights, the function*

$$\frac{w(x) + w(y)}{\text{mult}_w(f)}$$

is minimized by the choice such that the entire Newton polygon is contained in the half plane $w(x)i + w(y)j \geq \text{mult}_w f$, and the line $w(x)i + w(y)j = \text{mult}_w f$ contains the point where the Newton polygon intersects the diagonal line $j = i$.

Proof. See [8]. □

Lemma 1.2.4. *Let $f = x^{m_1}y^{n_1} + x^{m_2}y^{n_2}$, where m_1, m_2, n_1, n_2 are arbitrary nonnegative integers. Then*

$$c_0(f) = \min \left\{ \frac{m_1 - n_1 - m_2 + n_2}{m_1 n_2 - m_2 n_1}, \frac{1}{\gcd(m_1, m_2)}, \frac{1}{\gcd(n_1, n_2)} \right\}.$$

Proof. See [20]. □

2. LOG CANONICAL THRESHOLD OF A FANO HYPERSURFACE

2.1. A general hypersurface. As we mentioned, one can consider the first global log canonical threshold of a general hypersurface of degree $n + 1 \geq 4$ in \mathbb{P}^{n+1} in behalf of Conjecture 1.1.4.

Proposition 2.1.1. *Let X be a general hypersurface of degree $n + 1 \geq 4$ in \mathbb{P}^{n+1} . Then $\text{lct}_1(X) = 1$.*

Proof. We only consider general quartic threefolds. The cases for $n \geq 5$ have the same proofs.

Let X be a general smooth quartic hypersurface in \mathbb{P}^4 . To prove Proposition 2.1.1, we have to show that the log pair (X, H) is log canonical for each divisor H in $| -K_X |$.

Consider the space

$$\mathcal{S} = \mathbb{P}^4 \times H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(4))$$

with the natural projections $p: \mathcal{S} \rightarrow H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(4))$ and $q: \mathcal{S} \rightarrow \mathbb{P}^4$. Put

$$\mathcal{I} = \left\{ (O, F) \in \mathcal{S} \mid F(O) = 0 \text{ and } F = 0 \text{ is smooth.} \right\}$$

Let (O, F) be a pair in \mathcal{I} . Suppose that $O = [0 : 0 : 0 : 0 : 1]$. Then F can be given by an equation

$$w^3x + w^2q_2(x, y, z, t) + wq_3(x, y, z, t) + q_4(x, y, z, t) \in \mathbb{C}[x, y, z, t, w],$$

where q_i is a homogeneous polynomial of degree i .

We will say that the point O is *bad* on the hypersurface $F = 0$ if one of the following condition holds:

- (1) $q_2(0, y, z, t) = 0$.
- (2) $q_2(0, y, z, t) = l^2(y, z, w)$ for some linear form $l(y, z, t)$ and if we assume $l(y, z, t) = y$, the equation $q_3(0, 0, z, t) = 0$ defines a triple point in \mathbb{P}^1 .

Let Q be a point in \mathbb{P}^4 and Ω be the fiber of p over the point Q . Then $\dim(\Omega) = \dim(H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(4)))$. Put

$$\mathcal{Y} = \left\{ (O, F) \in \mathcal{I} \mid \text{the point } O \text{ is bad on the quartic } F = 0. \right\}$$

The restriction $p|_{\mathcal{Y}}: \mathcal{Y} \rightarrow \mathbb{P}^4$ is surjective. Moreover, elementary calculations show

$$\dim(\Omega \cap \mathcal{Y}) = \dim(\Omega) - 5,$$

which implies that the restriction

$$q|_{\mathcal{Y}}: \mathcal{Y} \rightarrow H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(4))$$

is not surjective. So, a general quartic threefold in \mathbb{P}^4 has no bad point.

Suppose that the log pair (X, H) is not log canonical. Then, the surface H is not log canonical as well (see Theorem 7.5 in [19]). The surface H has at most isolated singularities (see [17] or [28]), which implies that there is a point P of the quartic surface H at which the surface H is not log canonical. Hence, the surface H is cut out on the quartic X by the tangent hyperplane to the hypersurface X at the point P . We may assume that P is not a bad point on X . Then, it is easy to check that the surface H is log canonical at the point P , which is a contradiction. \square

2.2. Smooth quartic. In order to prove Theorem 1.1.5, we will investigate log canonical thresholds of normal quartic surfaces H in \mathbb{P}^3 because the log canonical thresholds of H in \mathbb{P}^3 and in X are the same (see [11]).

If a normal quartic surface H is smooth, then the log canonical threshold of H is 1. If a normal quartic surface H is a cone over a smooth quartic curve, then log canonical threshold of H is $\frac{3}{4}$. Furthermore, the converse is also true, that is, if the log canonical threshold of H is $\frac{3}{4}$, then H is a cone over a smooth quartic curve (see Example 1.1.3).

Let H be a normal surface in \mathbb{P}^3 defined by a homogeneous quartic polynomial F . We suppose that H has a singular point at $[0:0:0:1]$. We will then consider the log pair (\mathbb{C}^3, D) , where D is the fourth affine piece of H that is defined by the polynomial $f(x, y, z) = F(x, y, z, 1)$. Since log canonical thresholds can be computed locally, it is enough to study the log pair (\mathbb{C}^3, D) instead of (X, H) .

CASE A. Singularities of multiplicity 2.

Suppose that $f = 0$ has a singular point of multiplicity 2 at the origin. We may then write

$$f(x, y, z) = f_2(x, y, z) + f_3(x, y, z) + f_4(x, y, z),$$

where f_d is a homogeneous polynomial of degree d .

We consider the tangent cone $T_0 D$ of D at the point $(0, 0, 0)$. If it is irreducible, then $c_0(f_2) = 1$. And if it consists of two distinct planes, then we get $c_0(f_2) = 1$ too. In both the cases, we get $c_0(f) = 1$ by Lemma 1.2.1. Therefore, we may assume that the tangent cone $T_0 D$ of D is a double plane.

- By a linear coordinate change, we may assume that $f_2 = x^2$.

Now, we write

$$f_3(x, y, z) = g_3(y, z) + xg_2(y, z) + x^2g_1(y, z) + x^3g_0(y, z)$$

and

$$f_4(x, y, z) = h_4(y, z) + xh_3(y, z) + x^2h_2(y, z) + x^3h_1(y, z) + x^4h_0(y, z),$$

where g_l and h_m are homogeneous polynomials of degrees l and m .

At first, we consider weight $w = (3, 2, 2)$. It is easy to check that $(f_w = x^2 + g_3(y, z) = 0) \subset \mathbb{C}^3$ has an isolated singularity at the origin if g_3 is nonzero and reduced. In this case, the log canonical threshold $c_0(f) = 1$, and hence we may assume that $g_3(y, z) = y^2z, y^3$, or 0. Moreover, if $g_3 = y^2z$, then we can easily check that $c_0(f) = 1$. Consequently,

- we may assume that $g_3(y, z)$ is either y^3 or 0.

Note that if $g_3(y, z) = 0$, then $h_4(y, z) \neq 0$. If $g_3(y, z) = 0$ and h_4 is reduced, then we can see that $c_0(f) = 1$ using weight $w = (3, 1, 1)$. Therefore,

- if $g_3(y, z) = 0$, we may assume that $h_4(y, z) = y^4, y^3z$ or $y^2z(z + ay)$, where a is a constant.

Using holomorphic change of variables, we are to obtain a suitable holomorphic function for our computations, which is holomorphically equivalent to $f(x, y, z)$. In what follows we use the following notations:

- ρ_m is a homogenous polynomial of degree m in variables y and z ;
- φ_m is a homogenous polynomial of degree m in variables x, y , and z ;
- $\rho_{\geq m}$ is a polynomial with the lowest degree m in variables y and z ;
- $\varphi_{\geq m}$ is a polynomial with the lowest degree m in variables x, y , and z ;
- holomorphic equivalence is denoted by \sim .

Apply the coordinate transform to f as follows:

$$\begin{aligned} f(x, y, z) &\sim f\left(x - \frac{g_0}{2}x^2 - \frac{g_1}{2}x - \frac{g_2}{2}, y, z\right) \\ &= x^2 + g_3 + \left(h_4 - \frac{1}{4}g_2^2\right) + \rho_{\geq 5}(y, z) + x\varphi_{\geq 3}(x, y, z). \end{aligned}$$

Set

$$f(x, y, z) = x^2 + \psi_3(y, z) + \psi_4(y, z) + \rho_{\geq 5}(y, z) + x\varphi_3(x, y, z) + x\varphi_{\geq 4}(x, y, z),$$

where $\psi_3(y, z) = g_3(y, z)$ and $\psi_4(y, z) = h_4(y, z) - \frac{1}{4}g_2(y, z)^2$. Again, apply the coordinate transform to f as follows:

$$\begin{aligned} f(x, y, z) &\sim f\left(x - \frac{\varphi_3(x, y, z)}{2}, y, z\right) \\ &= x^2 + \psi_3(y, z) + \psi_4(y, z) + \psi_5(y, z) + \psi_6(y, z) + \\ &\quad + \rho_{\geq 7}(y, z) + x\varphi_4(x, y, z) + x\varphi_{\geq 5}(x, y, z), \end{aligned}$$

where $\psi_5(y, z) = \frac{1}{4}g_1(y, z)g_2(y, z)^2 - \frac{1}{2}g_2(y, z)h_3(y, z)$ and $\psi_6(y, z) = g_2(y, z)\zeta_4(y, z) - \frac{1}{4}h_3(y, z)^2$ for some homogeneous polynomial ζ_4 of degree 4 in variables y and z . Set

$$f(x, y, z) = x^2 + \psi_3(y, z) + \psi_4(y, z) + \psi_5(y, z) + \psi_6(y, z) + \rho_{\geq 7}(y, z) + x\varphi_4(x, y, z) + x\varphi_{\geq 5}(x, y, z).$$

Once again, apply the coordinate transform to f as follows:

$$\begin{aligned} f(x, y, z) &\sim f\left(x - \frac{\varphi_4(x, y, z)}{2}, y, z\right) \\ &= x^2 + \psi_3(y, z) + \psi_4(y, z) + \psi_5(y, z) + \psi_6(y, z) + \psi_7(y, z) + \psi_8(y, z) + \\ &\quad + \rho_{\geq 9}(y, z) + x\varphi_5(x, y, z) + x\varphi_{\geq 6}(x, y, z), \end{aligned}$$

where ψ_7 and ψ_8 are homogeneous polynomials of degrees 7 and 8 in variables y and z . It is easy to see that the homogeneous terms ψ_i , $i = 3, 4, \dots, 8$, remain unchanged in regardless of further transforms of the type $(x, y, z) \mapsto (x - \frac{\varphi_m(x, y, z)}{2}, y, z)$. In this way, for a sufficiently large N , we can obtain a holomorphically equivalent polynomial

$$h(x, y, z) = x^2 + \sum_{m=3}^{N-1} \psi_m(y, z) + \varphi_{\geq N}(x, y, z),$$

from the original quartic polynomial f . Put $g(y, z) = \sum_{m=3}^{N-1} \psi_m(y, z)$. Instead of the original quartic polynomial f , we can always use the polynomial h . Furthermore, we can easily see that the log canonical threshold $c_0(x^2 + g(y, z))$ is the same as $c_0(f)$ in our computations. Therefore, it suffices to see what values of log canonical threshold can appear for the polynomial $g(y, z)$ of two variables.

The normality of the quartic surface H gives the function $g(y, z)$ some restriction.

Lemma 2.2.1. *Suppose $z^2 \in g_2$. If the leading term of $g(y, z)$ contains $y^3 z^k$ for some $0 \leq k \leq 2$, then the Newton polygon for the function g contains the point corresponding to yz^6 .*

Proof. (We get the terms ψ_m , $m = 6, 7, 8$, with the aid of a computer program, *Mathematica*.)

Suppose that the Newton polygon for the function g does not contain the point corresponding to yz^6 . Then, we see that $z^4, z^3 y \notin h_4 - \frac{1}{4}g_2^2 = \psi_4$ and $z^5, z^4 y \notin \frac{1}{4}g_2(g_1 g_2 - 2h_3) = \psi_5$, which means that $h_4 = \frac{1}{4}g_2^2 + y^2\alpha$ and $g_1 g_2 - 2h_3 = y^2\beta$, where α and β are polynomials in variables y and z .

By replacing h_3 by $\frac{g_1 g_2 - y^2\beta}{2}$, we obtain

$$\psi_6 = g_2 \varphi_4 - \frac{1}{4}h_3^2 = \frac{1}{16}g_2^2 (4h_2 - 2g_0 g_2 - g_1^2) + y^2 \rho,$$

where ρ is a polynomial in variables y and z . Because $z^6, z^5 y \notin \psi_6$, we get $4h_2 - 2g_0 g_2 - g_1^2 = cy^2$, where c is a constant. With the identity $h_2 = \frac{1}{4}(2g_0 g_2 + g_1^2 + cy^2)$, we can obtain

$$\psi_7 = g_2^3 (g_0 g_1 - 2h_1) + y^2 \bar{\rho},$$

where $\bar{\rho}$ is a polynomial in variables y and z . Since $z^7, z^6 y \notin \psi_7$, we see that $g_1 g_2 - 2h_1 = 0$. With the identity $h_1 = \frac{g_0 g_1}{2}$, we can get

$$\psi_8 = \frac{1}{64}g_2^4 (4h_0 - g_0^2) + y^2 \tilde{\rho},$$

where $\tilde{\rho}$ is a polynomial in variables y and z . Because ψ_8 cannot contain z^8 , it must be zero, and hence $4h_0 = g_0^2$.

Look back to our original polynomial

$$\begin{aligned} f &= x^2 + ay^3 + x(g_2 + g_1 x + g_0 x^2) + \\ &\quad + \frac{1}{4}g_2^2 + y^2\alpha + x\left\{\frac{1}{4}g_0^2 x^3 + \frac{1}{2}g_0 g_1 x^2 + \left(\frac{1}{4}g_1^2 + \frac{1}{2}g_0 g_2 + \frac{c}{4}y^2\right)x - \frac{1}{2}y^2\beta + \frac{1}{2}g_1 g_2\right\} \\ &= \left\{\frac{1}{2}g_0 x^2 + \left(1 + \frac{1}{2}g_1\right)x + \frac{1}{2}g_2\right\}^2 + \frac{1}{4}y^2\phi_2(x, y, z), \end{aligned}$$

where $\phi_2(x, y, z)$ is a polynomial of degree 2 and $a = 0$ when the leading term of g is not y^3 . Then, we can see $\text{Sing}(f = 0) \supset (\frac{1}{2}g_0x^2 + (1 + \frac{1}{2}g_1)x + \frac{1}{2}g_2 = 0, y = 0)$. It cannot happen because the surface H is normal. \square

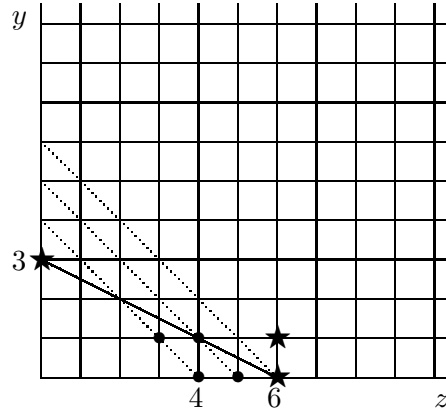
We now start computing the log canonical thresholds $c_0(f)$ at the origin case by case.

1. $g_3(y, z) = y^3$.

Note that if $y^2z^2 \in \psi_4$, the log canonical threshold is $c_0(f) = 1$. Therefore, we may assume $y^2z^2 \notin \psi_4$.

(a) $z^2 \notin g_2$.

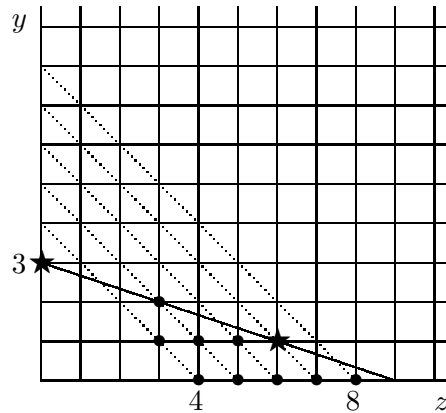
In the following diagram, the terms corresponding to \star must be contained in the Newton polygon of $g(y, z)$ by our conditions and Lemma 2.2.1. The terms corresponding to \bullet may or may not appear in the polynomial $g(y, z)$. However, the terms corresponding to blank lattices below the solid slant line cannot appear in the polynomial $g(y, z)$. Note that the normality of H implies $z^3 \in h_3$, and hence $z^6 \in g_2\zeta_4 - \frac{1}{4}h_3^2 = \psi_6$.



The diagram shows that the value $c_0(g_w)$ is at least $c_0(y^3 + z^6) = \frac{1}{2}$ by Lemma 1.2.3. Thus, the log canonical threshold $c_0(f) = 1$.

(b) $z^2 \in g_2$.

The following diagram uses the same notations as those in the previous diagram. The terms corresponding to \star must be contained in the Newton polygon of $g(y, z)$ by our conditions and Lemma 2.2.1.



Depending on whether the term corresponding to each \bullet is contained in $g(y, z)$ or not, we can get the following possible log canonical thresholds of f :

$$w = (9, 6, 2), \quad h_w = x^2 + y^3 + y^2z^3, \quad c_0(f) = \frac{17}{18};$$

$$\begin{aligned}
w &= (15, 10, 4), \quad h_w = x^2 + y^3 + yz^5, \quad c_0(f) = \frac{29}{30}; \\
w &= (9, 6, 2), \quad h_w = x^2 + y^3 + yz^6, \quad c_0(f) = \frac{17}{18}; \\
w &= (21, 14, 6), \quad h_w = x^2 + y^3 + z^7, \quad c_0(f) = \frac{41}{42}; \\
w &= (12, 8, 3), \quad h_w = x^2 + y^3 + z^8, \quad c_0(f) = \frac{23}{24}.
\end{aligned}$$

Note that the zero loci of all leading terms are log canonical in the outside of the origin.

2. $g_3 = 0$.

In this case, if the point corresponding to y^2z^2 is a vertex of the Newton polygon of g , then $c_0(f) = 1$. Therefore, we may assume that this is not a case. From the normality of H , we have either $z^2 \in g_2$ or $z^3 \in h_3$.

(a) $z^2 \notin g_2$

Note that the condition implies $yz^3, z^4 \notin \psi_4$ and $z^3 \in h_3$.

(1) $y^3z \in \psi_4$

If $zy \in g_2$, using weight $w = (3, 2)$, $g_w = y^3z + yz^4$. Because g_w has an isolated singularity at the origin, we get $c_0(g) = \frac{5}{11}$ from Lemma 1.2.4, and hence $c_0(f) = \frac{21}{22}$. If $zy \notin g_2$, use weight $w = (5, 3)$ to get $g_w = y^3z + z^6$. Then, g_w has an isolated singularity at the origin. Therefore, the log canonical threshold $c_0(g) = \frac{4}{9}$, and hence $c_0(f) = \frac{17}{18}$.

(2) $y^3z \notin \psi_4$ but $y^4 \in \psi_4$

If $zy \in g_2$, then use weight $w = (4, 3)$. We have $g_w = y^4 + yz^4$. Since $(g_w = 0) \subset \mathbb{C}^2$ has an isolated singularity at the origin, $c_0(g) = \frac{7}{16}$ by Lemma 1.2.4, and hence $c_0(f) = \frac{15}{16}$.

If $zy \notin g_2$, use weight $w = (3, 2)$ to get $g_w = y^4 + z^6$. Hence, $c_0(g) = \frac{5}{12}$ and $c_0(f) = \frac{11}{12}$.

(3) $\psi_4 = 0$

It follows from our assumption on h_4 that either $g_2 = 2yz$ or $g_2 = 2y^2$.

(i) $g_2 = 2yz$ and $h_4 = y^2z^2$.

The normality of H implies

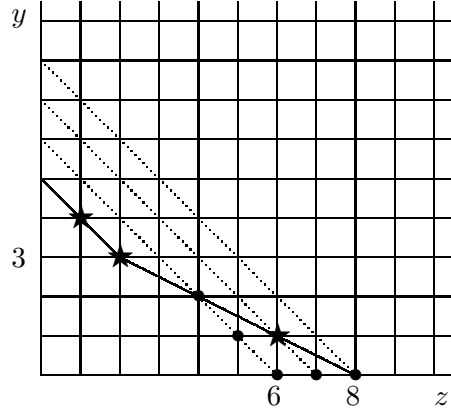
$$y^3, z^3 \in h_3;$$

$$\psi_5 = \frac{1}{4}g_1g_2^2 - \frac{1}{2}g_2h_3 = ayz(y^3 + by^2z + cyz^2 + dz^3), \text{ where } a, d \neq 0.$$

Note that if ψ_5 has no tripled factor, then $h_w = x^2 + \psi_5$ with weight $w = (5, 2, 2)$ and $(\mathbb{C}^3, h_w = 0)$ is log canonical in the outside of the origin. Thus the log canonical threshold $c_0(f) = \frac{9}{10}$.

Now assume $\psi_5 = a(y + bz)^3yz$, where $a, b \neq 0$. By an appropriate linear coordinate change, we can assume $g = y^4z - ey^3z^2 + (\text{higher terms})$, where $e \neq 0$. Note that the linear change makes g_2 contain the term z^2 .

The diagram below uses the same notations as those in the previous diagrams. The terms corresponding to \star must be contained in the Newton polygon of $g(y, z)$ by our conditions and Lemma 2.2.1.



Depending on whether the term corresponding to each \bullet is contained in $g(y, z)$ or not, we can get the following possible log canonical thresholds of f :

$$\begin{aligned} w = (4, 2, 1), h_w &= x^2 + y^3 z^2 + y^2 z^4, c_0(f) = \frac{7}{8}; \\ w = (13, 6, 4), h_w &= x^2 + y^3 z^2 + y z^5, c_0(f) = \frac{23}{26}; \\ w = (9, 4, 3), h_w &= x^2 + y^3 z^2 + z^6, c_0(f) = \frac{17}{18}; \\ w = (4, 2, 1), h_w &= x^2 + y^3 z^2 + y z^6, c_0(f) = \frac{7}{8}; \\ w = (21, 10, 6), h_w &= x^2 + y^3 z^2 + z^7, c_0(f) = \frac{37}{42}; \\ w = (4, 2, 1), h_w &= x^2 + y^3 z^2 + z^8, c_0(f) = \frac{7}{8}. \end{aligned}$$

(ii) $g_2 = 2y^2$ and $h_4 = y^4$.

Because $z^3 \in h_3$, if the equation $\psi_5 = 0$ has a triple point in \mathbb{P}^1 , then $\psi_5 = ay^2(z + by)^3$, where $a, b \neq 0$. By a linear coordinate change, we can assume that the leading term of g is $y^2 z^3$ and $y^2 \in g_2$. It is the same case as (i) if we exchange y with z .

(b) $z^2 \in g_2$

Note that $\psi_4 = 0$ cannot define a quadruple point in \mathbb{P}^1 . Also, if it defines no triple point, then $c_0(f) = 1$.

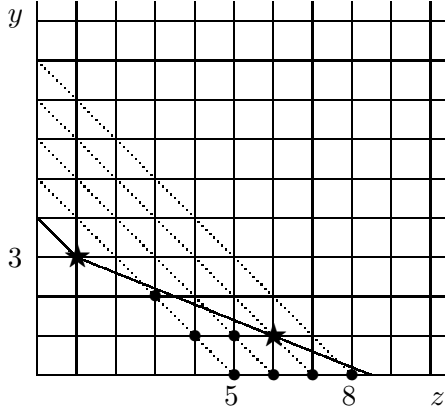
(1) $h_4 = y^4$

The polynomial ψ_4 can define at most a double point in \mathbb{P}^1 . Therefore, $c_0(f) = 1$.

(2) $h_4 = y^3 z$ or $h_4 = y^2 z(z + ay)$, where a is a constant.

By an appropriate coordinate change, we may assume that $\psi_4 = y^3 z$. Moreover, the polynomial g_2 still contains z^2 after the linear coordinate transform.

In the following diagram, the terms corresponding to \star must be contained in the Newton polygon of $g(y, z)$ by our conditions and Lemma 2.2.1. The other notations are the same as before.



Depending on whether the term corresponding to each \bullet is contained in $g(y, z)$ or not, we can get the following possible log canonical thresholds of f :

$$\begin{aligned}
w = (7, 4, 2), \quad h_w &= x^2 + y^3z + y^2z^3, \quad c_0(f) = \frac{13}{14}; \\
w = (11, 6, 4), \quad h_w &= x^2 + y^3z + yz^4, \quad c_0(f) = \frac{21}{22}; \\
w = (15, 8, 6), \quad h_w &= x^2 + y^3z + z^5, \quad c_0(f) = \frac{29}{30}; \\
w = (7, 4, 2), \quad h_w &= x^2 + y^3z + yz^5, \quad c_0(f) = \frac{13}{14}; \\
w = (9, 5, 3), \quad h_w &= x^2 + y^3z + z^6, \quad c_0(f) = \frac{17}{18}; \\
w = (17, 10, 4), \quad h_w &= x^2 + y^3z + yz^6, \quad c_0(f) = \frac{31}{34}; \\
w = (7, 4, 2), \quad h_w &= x^2 + y^3z + z^7, \quad c_0(f) = \frac{13}{14}; \\
w = (12, 7, 3), \quad h_w &= x^2 + y^3z + z^8, \quad c_0(f) = \frac{11}{12}.
\end{aligned}$$

CASE B. Singularities of multiplicity 3.

Now, we suppose that $f = 0$ has a singular point of multiplicity 3 at the origin. We may then write

$$f(x, y, z) = f_3(x, y, z) + f_4(x, y, z),$$

where f_d is a homogeneous polynomial of degree d .

We consider the plane cubic curve $C \subset \mathbb{P}^2$ defined by f_3 . It follows from Lemma 1.2.1 that the log pair (\mathbb{P}^2, C) is log canonical if and only if $c_0(f) = 1$. Therefore,

- we may assume that $f_3(x, y, z) = x^3 + y^3, x^2y + y^2z, x^2z + y^3, x^2y$, or x^3 .

By simple computation, we can check that f_4 contains at least one of monomials z^4, z^3x , and z^3y , which follows from the normality of H .

1. $f_3 = x^3 + y^3$.

- (a) $z^4 \in f_4$.

Using weight $w = (4, 4, 3)$, we get the leading term $f_w = x^3 + y^3 + z^4$. Since $(f_w = 0) \subset \mathbb{C}^3$ has an isolated singularity at the origin, the log canonical threshold $c_0(f) = \frac{11}{12}$.

- (b) $z^4 \notin f_4$.

Use weight $w = (3, 3, 2)$. Then, we get the leading term $f_w = x^3 + y^3 + az^3x + bz^3y$, where a and b are constants. Note that one of a and b is not zero. Then $(f_w = 0) \subset \mathbb{C}^3$ has an isolated singularity at the origin. Therefore, the log canonical threshold $c_0(f) = \frac{8}{9}$.

2. $f_3 = x^2y + y^2z$.

- (a) $z^4 \in f_4$.

Use weight $w = (5, 6, 4)$. Then $f_w = x^2y + y^2z + az^4$ with a nonzero constant a . And $(f_w = 0) \subset \mathbb{C}^3$ has an isolated singularity at the origin. Therefore, $c_0(f) = \frac{15}{16}$.

- (b) $z^4 \notin f_4$ and $z^3x \notin f_4$.
 Use weight $w = (3, 4, 2)$ to get the leading term $f_w = x^2y + y^2z + ayz^3$, where a is a nonzero constant. It is easy to check that log pair $(\mathbb{C}^3, (f_w = 0))$ is log canonical outside of the origin. Thus, $c_0(f) = \frac{9}{10}$.
- (c) $z^4 \notin f_4$ and $z^3x \in f_4$.
 Using weight $w = (4, 5, 3)$, we get the leading term $f_w = x^2y + y^2z + axz^3$, where a is a nonzero constant. Since $(f_w = 0) \subset \mathbb{C}^3$ has an isolated singularity at the origin, the log canonical threshold $c_0(f) = \frac{12}{13}$.
3. $f_3 = x^2z + y^3$
- (a) $z^4 \in f_4$.
 Use weight $w = (9, 8, 6)$. Then $f_w = x^2z + y^3 + az^4$ with a nonzero constant a . And $(f_w = 0) \subset \mathbb{C}^3$ is log canonical in the outside of the origin. Therefore, $c_0(f) = \frac{23}{24}$.
- (b) $z^4 \notin f_4$ and $z^3y \notin f_4$.
 Using weight $w = (6, 5, 3)$, we get the leading term $f_w = x^2z + y^3 + axz^3$, where a is a nonzero constant. Since $(f_w = 0) \subset \mathbb{C}^3$ has an isolated singularity at the origin, the log canonical threshold $c_0(f) = \frac{14}{15}$.
- (c) $z^4 \notin f_4$ and $z^3y \in f_4$.
 Using weight $w = (7, 6, 4)$, we get the leading term $f_w = x^2z + y^3 + ayz^3$, where a is a nonzero constant. Since $(f_w = 0) \subset \mathbb{C}^3$ has an isolated singularity at the origin, the log canonical threshold $c_0(f) = \frac{17}{18}$.
4. $f_3 = x^2y$.
 Use weight $w = (3, 2, 2)$. Then, we have the leading term $f_w = x^2y + g(y, z)$, where g is a quartic homogeneous polynomial. We claim that if $g(y, z)$ is a nonzero reduced polynomial, the log pair $(\mathbb{C}^3, (f_w = 0))$ is log canonical in the outside of the origin, and hence the log canonical threshold $c_0(f) = \frac{7}{8}$. To prove the claim, write

$$g(y, z) = ay^4 + by^3z + cy^2z^2 + dyz^3 + ez^4.$$

If $e \neq 0$, then $(f_w = 0)$ has an isolated singularity at the origin. If $e = 0$, then d cannot be zero and $\text{Sing}(f_w = 0) = (y = 0, x^2 + dz^3)$. Consider $f_w(x + t^3, y, z - \zeta t^2) = 2t^3xy + d\zeta^2t^2yz + c\zeta^2t^2y^2 + (\text{higher order terms})$, where $\zeta^3 = -\frac{1}{d}$. Since $c_0(2t^3xy + d\zeta^2t^2yz + c\zeta^2t^2y^2) = 1$ for $t \neq 0$, the log canonical pair $(\mathbb{C}^3, (f_w = 0))$ is log canonical in the outside of the origin.

Now, we suppose that $g(y, z)$ is non-reduced. Then, we may have the following cases by appropriate linear coordinate changes.

- (a) $g(y, z) = z^4$.
 By the normality of H , f_4 contains xy^3 . Use weight $w = (8, 4, 5)$. Then we get $f_w = x^2y + z^4 + xy^3$. Since $(f_w = 0)$ has an isolated singularity at the origin, the log canonical threshold $c_0(f) = \frac{17}{20}$.
- (b) $g(y, z) = z^3(y + az)$, where a is constant.
 In this case, f_4 contains xy^3 too. Use weight $w = (6, 3, 4)$. Then, $f_w = x^2y + z^3y + bxy^3$, where b is a nonzero constant. Note that $\text{Sing}(f_w = 0) = (y = 0, x^2 + z^3 = 0)$. Consider $f_w(x + t^3, y, z - t^2) = (2t^3 + 3t^4)xy + (\text{higher order terms})$. Since $c_0((2t^3 + 3t^4)xy) = 1$ for $t \neq 0$, the log pair $(\mathbb{C}^3, (f_w = 0))$ is log canonical in the outside of the origin. Therefore, $c_0(f) = \frac{13}{15}$.
- (c) $g(y, z) = y^2z^2 + ayz^3 + bz^4$, where $a \neq 0, b \neq 0$ and $a^2 - 4b \neq 0$.
 Use weight $w = (3, 2, 2)$. Then $f_w = x^2y + y^2z^2 + ayz^3 + bz^4$. The singular locus of $(f_w = 0)$ is $(x = 0, z = 0)$. Considering $f_w(x, y + t, z) = tx^2 + t^2z^2 + (\text{higher order terms})$, we get $c_0(f) = \frac{7}{8}$.
- (d) $g(y, z) = y^2z^2$.
 Use weight $w = (5, 4, 3)$. Then $f_w = x^2y + y^2z^2 + axz^3$, where a is a nonzero

constant. Note that f_4 contains at least one of monomials z^4 , z^3x , and z^3y . We can easily check that the log pair $(\mathbb{C}^3, (f_w = 0))$ is log canonical in the outside of the origin. Therefore, log canonical threshold $c_0(f) = \frac{6}{7}$.

- (e) $g(y, z) = z^2(y + az)^2$, where a is a nonzero constant.
Use weight $w = (3, 2, 2)$. Then $f_w = x^2y + g(y, z)$. Note $\text{Sing}(f_w = 0) = (x = 0, z = 0)$. We can see that the log pair $(\mathbb{C}^3, (f_w = 0))$ is log canonical in the outside of the origin. Therefore, the log canonical threshold $c_0(f)$ is $\frac{7}{8}$.
- (f) $g(y, z) = y^2z^2 + ayz^3$, where a is a nonzero constant.
Use weight $w = (3, 2, 2)$. Then $f_w = x^2y + y^2z^2 + ayz^3$. Since $\text{Sing}(f_w = 0) = (x = 0, z = 0) \cup (y = 0, x^2 + az^3 = 0)$, we consider $f_w(x, y + t, z) = tx^2 + t^2z^2 + (\text{higher order terms})$ and $f_w(x + s^3, y, z + \xi s^2) = 2s^3xy + \xi^2s^4y^2 + 3a\xi^2yz + (\text{higher order terms})$, where $\xi^3 = -\frac{1}{a}$. And we see that $c_0(tx^2 + t^2z^2) = 1$ and $c_0(2s^3xy + \xi^2s^4y^2 + 3a\xi^2yz) = 1$ for $t \neq 0$ and $s \neq 0$. Therefore, $c_0(f) = \frac{7}{8}$.
- (g) $g(y, z) = y^2z^2 + az^4$, where a is a nonzero constant.
Use weight $w = (3, 2, 2)$. Then $f_w = x^2 + y^2z^2 + az^4$. Note that $\text{Sing}(f_w = 0) = (x = 0, z = 0)$. Consider $f_w(x, y + t, z) = tx^2 + t^2z^2 + (\text{higher order terms})$. Since $c_0(tx^2 + t^2z^2) = 1$ for $t \neq 0$, the log pair $(\mathbb{C}^3, (f_w = 0))$ is log canonical in the outside of the origin. Therefore, $c_0(f) = \frac{7}{8}$.
- (h) $g(y, z) = y^4$.
Note that f_4 contains xz^3 . Using weight $w = (9, 6, 5)$, we get $f_w = x^2y + y^4 + xz^3$. Since $(f_w = 0)$ has an isolated singularity at the origin, the log canonical threshold $c_0(f) = \frac{5}{6}$.
- (i) $g(y, z) = y^3z$.
In this case, f_4 also contains xz^3 . Use weight $w = (7, 5, 4)$ and we get $f_w = x^2y + y^3z + axz^3$, where a is a nonzero constant. Then, we can easily check that $(f_w = 0)$ has an isolated singularity at the origin. Thus, the log canonical threshold $c_0(f) = \frac{16}{19}$.
- (j) $g(y, z) = y^2z(z + ay)$, where a is a constant.
The polynomial f_4 contains xz^3 again. Using weight $w = (5, 4, 3)$, we get $f_w = x^2y + y^2z^2 + bxz^3$. Then we can check that $\text{Sing}(f_w = 0) = (x = 0, z = 0)$ plus one isolated singularity. Consider $f_w(x, y + t, z) = tx^2 + t^2z^2 + (\text{higher order terms})$. Since $c_0(tx^2 + t^2z^2) = 1$ for $t \neq 0$, the log pair $(\mathbb{C}^3, (f_w = 0))$ is log canonical in the outside of the origin along $(x = 0, z = 0)$. Therefore, $c_0(f) = \frac{6}{7}$.

5. $f_3 = x^3$.

Using weight $w = (4, 3, 3)$, we get the leading term $f_w = x^3 + g(y, z)$, where g is a quartic homogeneous polynomial. If $g(y, z)$ is a nonzero reduced polynomial, then $(f_w = 0) \subset \mathbb{C}^3$ has an isolated singularity at the origin and hence $c_0(f) = \frac{5}{6}$.

Suppose that $g(y, z)$ is non-reduced. Then we may assume that $g(y, z) = z^4, yz^3, y^2z^2$, or $yz^2(y + z)$.

- (a) $g(y, z) = z^4$.
In this case, f_4 always contains xy^3 . Use weight $w = (12, 8, 9)$. Then we get $f_w = x^3 + z^4 + xy^3$. Since $(f_w = 0) \subset \mathbb{C}^3$ has an isolated singularity at the origin, $c_0(f) = \frac{29}{36}$.
- (b) $g(y, z) = yz^3$.
The polynomial f_4 contains xy^3 . Use weight $w = (9, 6, 7)$. Then we get $f_w = x^3 + z^4 + xy^3$. Since $(f_w = 0) \subset \mathbb{C}^3$ has an isolated singularity at the origin, $c_0(f) = \frac{22}{27}$.
- (c) $g(y, z) = y^2z^2$.
Use weight $w = (4, 3, 3)$, and get $f_w = x^3 + y^2z^2$. Note that $\text{Sing}(f_w = 0) = (x = 0, yz = 0)$. Consider $f_w(x, y + t, z) = t^2z^2 + 2tyz^2 + x^3 + y^2z^2$, $f_w(x, y, z + s) =$

$s^2y^2 + 2syz + x^3 + y^2z^2$. Then we have $c_0(t^2z^2 + 2tyz^2 + x^3) = \frac{5}{6}$ for $t \neq 0$, and $c_0(s^2y^2 + 2syz + x^3) = \frac{5}{6}$ for $s \neq 0$. Therefore, the log pair $(\mathbb{C}^3, \frac{5}{6}(f_w = 0))$ is log canonical in the outside of the origin. Consequently, $c_0(f) = \frac{5}{6}$.

(d) $g(y, z) = y^2z^2 + yz^3$.

Use weight $w = (4, 3, 3)$. Then we get $f_w = x^3 + y^2z^2 + yz^3$ and $\text{Sing}(f_w = 0) = (x = 0, y = 0)$. Consider $f_w(x, y, z + t) = t^2y^2 + 2ty^2z + ty^3 + x^3 + z^4$. Since $c_0(t^2y^2 + 2ty^2z + ty^3 + x^3) = \frac{5}{6}$ for $t \neq 0$, the log pair $(\mathbb{C}^3, \frac{5}{6}(f_w = 0))$ is log canonical in the outside of the origin. Therefore, $c_0(f) = \frac{5}{6}$.

So far, we have seen all the possible log canonical thresholds $c(X, H)$ for a hyperplane section H of X . The following two tables show that for each number μ of the set in Theorem 1.1.5, there exists a normal quartic surface D in \mathbb{P}^3 with $c(\mathbb{P}^3, D) = \mu$.

Multiplicity 2 at [0:0:0:1]	
μ	Equation
$\frac{37}{42}$	$2x^2w^2 + 2y^2z^2 + 4xyzw - xy^3 - xz^3 + 3x^2yw + 3x^2zw + 2x^3z$
$\frac{7}{8}$	$16x^2w^2 + 16y^2z^2 + 32xyzw - 8xy^3 - 8xz^3 + 24x^2yw + 24x^2zw + 9x^2(y + z)^2$
$\frac{9}{10}$	$x^2w^2 + y^2z^2 + 2xyzw + xy^3 + xz^3 + x^4$
$\frac{23}{26}$	$2x^2w^2 + 2y^2z^2 + 4xyzw - xy^3 - xz^3 + 3x^2yw + 3x^2zw + 2x^2z^2 + 2x^2yz$
$\frac{11}{12}$	$x^2w^2 + y^4 + xz^3 + x^4$
$\frac{13}{14}$	$* 81x^2w^2 + 81y^3z - 27y^2z^2 + 162xyzw - 48xz^2w + xy^2z - 72xyz^2 + 16xz^3 + 81x^4$
$\frac{15}{16}$	$x^2w^2 + y^2z^2 + 2y^3z + 2xy^2w + 2xyzw + xz^3 + x^4$
$\frac{31}{34}$	$* 27x^2w^2 + 27y^3z - 9y^2z^2 + 54xyzw - 16xz^2w + 27x^3y - 12x^3z$
$\frac{17}{18}$	$x^2w^2 + y^3z + xy^2w + xz^3 + x^4$
$\frac{21}{22}$	$x^2w^2 + y^3z + y^2z^2 + 2xyzw + xz^3 + x^4$
$\frac{23}{24}$	$x^2w^2 + y^3w + z^4 + 2xz^2w + x^3y + x^4$
$\frac{29}{30}$	$x^2w^2 + y^3w + z^4 + 2xz^2w + x^2yz + x^4$
$\frac{41}{42}$	$x^2w^2 + y^3w + z^4 + 2xz^2w + x^3z$

(*) These surfaces have another singularity at [0:0:1:0] with the value of log canonical threshold 1.

Multiplicity 3 at [0:0:0:1]			
μ	Equation	μ	Equation
$\frac{29}{36}$	$x^3w + z^4 + xy^3$	$\frac{22}{27}$	$x^3w + yz^3 + xy^3$
$\frac{5}{6}$	$x^2yw + xz^3 + x^4 + y^4$	$\frac{16}{19}$	$x^2yw + y^3z + xz^3 + x^4$
$\frac{17}{20}$	$x^2yw + xy^3 + x^4 + z^4$	$\frac{6}{7}$	$x^2yw + y^2z^2 + xz^3 + x^4 + xy^3$
$\frac{13}{15}$	$x^2yw + yz^3 + xy^3 + x^4$	$\frac{7}{8}$	$x^2yw + y^2z^2 + yz^3 + xy^3 + x^4 + z^4$
$\frac{8}{9}$	$x^3w + y^3w + xz^3 + yz^3 + x^4 + y^4$	$\frac{9}{10}$	$x^2yw + y^2zw + yz^3 + x^4 + y^4$
$\frac{11}{12}$	$x^3w + y^3w + x^4 + y^4 + z^4$	$\frac{12}{13}$	$x^2yw + y^2zw + xz^3 + x^4 + y^4$
$\frac{14}{15}$	$x^2zw + y^3w + xz^3 + x^4 + y^4$	$\frac{15}{16}$	$x^2yw + y^2zw + x^4 + y^4 + z^4$
$\frac{17}{18}$	$x^2zw + y^3w + yz^3 + x^4 + y^4$	$\frac{23}{24}$	$x^2zw + y^3w + x^4 + y^4 + z^4$

With the two tables above, the following lemma proves the second statement of Theorem 1.1.5.

Lemma 2.2.2. *Let $f_4(x, y, z, w)$ be a quartic homogeneous polynomial that defines a normal hypersurface in \mathbb{P}^3 . For a general cubic homogenous polynomial $g_3(x, y, z, w, t)$, the polynomial*

$$F(x, y, z, w, t) = f_4(x, y, z, w) + tg_3(x, y, z, w, t)$$

defines a smooth quartic 3-fold X in \mathbb{P}^4 with $\text{lct}_1(X) = c(X, D)$, where D is the hyperplane section of X defined by the equation $t = 0$.

Proof. Let

$$\mathcal{S} = \left\{ ((a_1, a_2, a_3, a_4), G) \in \mathbb{C}^4 \times H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(3)) \mid \begin{array}{l} f_4(a_1, a_2, a_3, a_4) + G(a_1, a_2, a_3, a_4, 1) = 0 \\ f_4 + tG = 0 \text{ is smooth.} \end{array} \right\}$$

with the natural projections $p : \mathcal{S} \rightarrow H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(3))$ and $q : \mathcal{S} \rightarrow \mathbb{C}^4$.

Note that, for a point $p = [a : b : c : d : 1]$, the coordinate change $[x : y : z : w : t] \mapsto [x - at : y - bt : z - ct : w - dt : t]$ transforms the homogeneous polynomial $f_4 + tG$ into

$$\begin{aligned} F &= f_4(x + at, y + bt, z + ct, w + dt) + tg_3(x + at, y + bt, z + ct, w + dt, t) \\ &= f_4(x, y, z, w) + t(G(x + at, y + bt, z + ct, w + dt, t) + H(x, y, z, w, t)), \end{aligned}$$

where H is a homogeneous cubic polynomial. It shows that under the coordinate change, the polynomial $f_4 + tG$ is transformed into the same type of polynomial, i.e., $f_4 + t\bar{G}$, where $\bar{G} \in H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(3))$.

For a pair $((a_1, a_2, a_3, a_4), G)$ in \mathcal{S} , suppose that the point (a_1, a_2, a_3, a_4) is the origin, and the tangent hyperplane to the quartic threefold $f_4 + tG = 0$ at the point $[0 : 0 : 0 : 0 : 1]$ is defined by $x = 0$. The quartic homogeneous polynomial $F = f_4 + tG$ can be given by an equation

$$t^3x + t^2q_2(x, y, z, w) + tq_3(x, y, z, w) + f_4(x, y, z, w),$$

where each q_i is a homogeneous polynomial of degree i and it is completely determined by G . We will say that the point $[a_1 : a_2 : a_3 : a_4 : 1]$ is *bad* on the hypersurface $F = 0$ if at least one of the following condition holds:

- (1) $q_2(0, y, z, w) = 0$.
- (2) $q_2(0, y, z, w) = l^2(y, z, w)$, where $l(y, z, w)$ is a linear form and if we assume $l(y, z, w)$ is y , $q_3(0, 0, z, w) = 0$ defines a triple point in \mathbb{P}^1 .

Put

$$\mathcal{Y} = \{((a_1, a_2, a_3, a_4), G) \in \mathcal{S} \mid [a_1 : a_2 : a_3 : a_4 : 1] \text{ is bad on the quartic } f_4 + tG = 0.\}$$

The restriction $p|_{\mathcal{Y}} : \mathcal{Y} \rightarrow \mathbb{C}^4$ is surjective. Let Q be a point in \mathbb{C}^4 and Ω be the fiber of p over the point Q . Then, as in Proposition 2.1.1, we can see

$$\dim(\Omega \cap \mathcal{Y}) = \dim(\Omega) - 5,$$

which implies that the restriction

$$q|_{\mathcal{Y}} : \mathcal{Y} \rightarrow H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(3))$$

is not surjective. Therefore, for a point (a_1, a_2, a_3, a_4) in \mathbb{C}^4 , the point $[a_1 : a_2 : a_3 : a_4 : 1]$ is not bad on the quartic $F = f_4 + tG = 0$ for a general G . Then, the similar argument as in Proposition 2.1.1 completes the proof. \square

2.3. General quartic. Let X be a smooth quartic hypersurface in \mathbb{P}^4 such that the following general conditions hold:

- the threefold X is 3-regular in the sense of the paper [29];
- every line on the hypersurface X has normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}$;
- the intersection of X with a two-dimensional linear subspace of \mathbb{P}^4 cannot be a double conic curve.

Remark 2.3.1. Every line on the quartic X has normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}$ if and only if no two-dimensional linear subspace of \mathbb{P}^4 is tangent to the quartic X along a line (see [6]).

Remark 2.3.2. It follows from the proof of Theorem 1.1.5 that $\text{let}_1(X) \geq \frac{16}{21}$.

Remark 2.3.3. Let B and B' be effective \mathbb{Q} -Cartier \mathbb{Q} -divisors on a variety V . Then

$$(V, \alpha B + (1 - \alpha) B')$$

is log canonical if both (V, B) and (V, B') are log canonical, where $0 \leq \alpha \leq 1$.

Let us prove Theorem 1.1.6. Put $\lambda = \frac{16}{21}$. Let D be an effective \mathbb{Q} -divisor on X such that $D \equiv -K_X$. To prove Theorem 1.1.6, we have to show that $(X, \lambda D)$ is log canonical.

Suppose that $(X, \lambda D)$ is not log canonical. Due to Remarks 2.3.2 and 2.3.3, we may assume that $D = \frac{1}{n}R$ where R is an irreducible divisor with $R \sim -nK_X$ for some natural number $n > 1$.

Lemma 2.3.4. *There is a point $P \in X$ such that $(X, \lambda D)$ is log canonical on $X \setminus P$.*

Proof. The log pair $(X, \lambda D)$ is log canonical in the outside of finitely many points of the quartic threefold X due to Theorem 2 in [28]. The required assertion follows from the fact that

$$H^1(X, \mathcal{J}(\lambda D)) = 0$$

by the Nadel vanishing (see [19]), where $\mathcal{J}(\lambda D)$ is the multiplier ideal sheaf of $(X, \lambda D)$. \square

Therefore, the log pair $(X, \lambda D)$ is not log canonical only at the point P .

The threefold X can be given by

$$w^3x + w^2q_2(x, y, z, t) + wq_3(x, y, z, t) + q_4(x, y, z, t) = 0 \subset \mathbb{P}^4 \cong \text{Proj}(\mathbb{C}[x, y, z, t, w]),$$

where $q_i(x, y, z, t)$ is a homogeneous polynomial of degree i . Furthermore, we may assume that the point P is located at $[0 : 0 : 0 : 0 : 1]$. Let T be the surface on X cut out by $x = 0$.

Lemma 2.3.5. *The multiplicity of D at the point P is at most 2.*

Proof. The statement immediately follows from the inequalities

$$4 = H \cdot T \cdot D \geq \text{mult}_P(T \cdot D) \geq \text{mult}_P(T) \text{mult}_P(D) \geq 2\text{mult}_P(D),$$

where H is a general hyperplane section of X passing through the point P . \square

Let $\pi: U \rightarrow X$ be the blow up at the point P and E be the π -exceptional divisor. Then

$$\bar{D} \equiv \pi^*(D) - \text{mult}_P(D) E,$$

where \bar{D} is the proper transform of the divisor D via the morphism π . The exceptional divisor E is isomorphic to \mathbb{P}^2 .

It follows from Corollary 3.5 in [7] or Proposition 3 in [30] that there is a line $L \subset E$ such that

$$\text{mult}_P(D) + \text{mult}_L(\bar{D}) > \frac{2}{\lambda}.$$

Let \mathcal{L} be the linear system of hyperplane sections of X such that

$$S \in \mathcal{L} \iff \text{either } L \subset \bar{S} \text{ or } S = T,$$

where \bar{S} is the proper transform of S via the birational morphism π . There is a two-dimensional linear subspace $\Pi \subset \mathbb{P}^4$ such that the base locus of \mathcal{L} consists of the intersection $\Pi \cap X$.

Let S be a general surface in \mathcal{L} . Then S is a smooth K3 surface and our general conditions imply that the curve $X|_\Pi$ is reduced (see Remark 2.3.1). Put

$$X|_\Pi = Z = \sum_{i=1}^r Z_i,$$

where each Z_i is an irreducible curve. Then $\sum_{i=1}^r \deg(Z_i) = 4$. It follows that

$$\text{mult}_P(S \cdot D) \geq \text{mult}_P(S) \text{mult}_P(D) + \text{mult}_L(\bar{S} \cdot \bar{D}) \geq \text{mult}_P(D) + \text{mult}_L(\bar{D}) > \frac{2}{\lambda},$$

where \bar{S} is the proper transform of the surface S via the birational morphism π .

Put

$$T_S = T|_S; \quad D_S = D|_S = \sum_{i=1}^r m_i Z_i + \Delta,$$

where m_i is a non-negative rational number and Δ is an effective one-cycle on S whose support does not contain the curves Z_1, \dots, Z_r . Then

$$\sum_{i=1}^r m_i \text{mult}_P(Z_i) + \text{mult}_P(\Delta) = \text{mult}_P(D_S) > \frac{2}{\lambda},$$

and the support of the cycle Δ does not contain any component of the cycle T_S . We have

$$4 = T_S \cdot D_S = \sum_{i=1}^r m_i \deg(Z_i) + T_S \cdot \Delta \geq \sum_{i=1}^r m_i \deg(Z_i) + 2 \left(\frac{2}{\lambda} - \sum_{i=1}^r m_i \text{mult}_P(Z_i) \right),$$

because $\text{mult}_P(T_S) \geq \text{mult}_P(T) = 2$. Note that $\text{mult}_P(Z) \geq 2$ implies $T_S = Z$.

Remark 2.3.6. The equality $m_i = \text{mult}_{Z_i}(D)$ holds for every i because $X|_{\Pi}$ is reduced.

It follows from the 3-regularity of X that $\text{mult}_P(Z) \leq 3$.

Lemma 2.3.7. *Suppose that $\text{mult}_P(Z) = 3$. Then*

$$16 > \frac{12}{\lambda} + \deg(Z_k) m_k,$$

where Z_k is the unique curve among Z_1, \dots, Z_r such that $\deg(Z_k) \geq 2$ or $P \notin Z_k$.

Proof. Let \bar{T} be the proper transform of the surface T via the birational morphism π . Then

$$3 = \text{mult}_P(Z) = \text{mult}_P(T \cdot S) = \text{mult}_P(T) \text{mult}_P(S) + \text{mult}_L(\bar{T} \cdot \bar{S}).$$

Hence, we see that $L \subset \bar{T}$. Since $\text{mult}_P(D) > \frac{1}{\lambda}$ and $\text{mult}_P(T) = 2$, it follows that

$$\text{mult}_P(T \cdot D) \geq \text{mult}_P(T) \text{mult}_P(D) + \text{mult}_L(\bar{T} \cdot \bar{D}) \geq 2 \text{mult}_P(D) + \text{mult}_L(\bar{D}) > \frac{3}{\lambda}.$$

Let L_1, \dots, L_m be all lines on X that pass through the point P . Put

$$T \cdot D = \sum_{i=1}^r \epsilon_i L_i + \bar{m}_k Z_k + \Upsilon,$$

where ϵ_i and \bar{m}_k are non-negative rational numbers, and Υ is an effective one-cycle on X whose support does not contain the lines L_1, \dots, L_m . Then $\bar{m}_k \geq m_k$ by Remark 2.3.6.

Let \mathcal{M} be a linear subsystem of the linear system $|-3K_X|$ such that

$$M \in \mathcal{M} \iff \text{mult}_P(M) \geq 4,$$

and let M be a general surface in \mathcal{M} . Then, the linear system \mathcal{M} contains surfaces cut by

$$xp_2(x, y, z, t) + (wx + q_2(x, y, z, t))p_1(x, y, z, t) + \mu(w^2x + wq_2(x, y, z, t) + q_3(x, y, z, t)) = 0,$$

where $p_i(x, y, z, t)$ is a homogeneous polynomial of degree i and $\mu \in \mathbb{C}$. Hence, the base locus of \mathcal{M} consists of the lines L_1, \dots, L_m due to the 3-regularity of the threefold X .

Taking the intersection with a general hyperplane section of X , we see that

$$4 \geq \sum_{i=1}^r \epsilon_i + \bar{m}_k \deg(Z_k),$$

but $\bar{m}_k \text{mult}_P(Z_k) + \text{mult}_P(\Upsilon) > \frac{3}{\lambda} - \sum_{i=1}^r \epsilon_i$. Hence, we have

$$\begin{aligned} 12 &= M \cdot T \cdot D \geq 3 \sum_{i=1}^r \epsilon_i + M \cdot (\bar{m}_k Z_k + \Upsilon) \\ &> 3 \sum_{i=1}^r \epsilon_i + 4 \left(\frac{3}{\lambda} - \sum_{i=1}^r \epsilon_i \right) = \frac{12}{\lambda} - \sum_{i=1}^r \epsilon_i, \end{aligned}$$

which implies $16 > 12/\lambda + \deg(Z_k)m_k$ because $4 \geq \sum_{i=1}^r \epsilon_i + \bar{m}_k \deg(Z_k) \bar{m}_k \geq m_k$. \square

From now on, in order to describe the reduced curve $Z = X \cap \Pi$, we will use the following notations:

- C : an irreducible cubic not passing through the point P .
- \tilde{C} : an irreducible cubic that is smooth at the point P .
- \hat{C} : an irreducible cubic that is singular at the point P .

For $i = 1, 2$

- Q_i : an irreducible quadric not passing through the point P .
- \tilde{Q}_i : an irreducible quadric passing through the point P .

For $i = 1, 2, 3, 4$

- L_i : a line not passing through the point P .
- \tilde{L}_i : a line passing through the point P .

Then, the following are all the possible configuration of Z . In each case, we derive a contradictory inequality from our assumptions so that the log pair $(X, \lambda D)$ should be log canonical.

CASE A. The curve Z is an irreducible quartic curve.

(1) $\text{mult}_P(Z) \leq 2$.

$$D_S = mZ + \Delta.$$

A contradictory inequality:

$$4 = T \cdot D \cdot S > 4m + 2 \left(\frac{2}{\lambda} - 2m \right) = \frac{4}{\lambda} > 4.$$

(2) $\text{mult}_P(Z) = 3$.

$$D_S = mZ + \Delta.$$

An auxiliary inequality:

$$16 > \frac{12}{\lambda} + 4m \quad \text{by Lemma 2.3.7.}$$

A contradictory inequality:

$$\begin{aligned} 4 &= T \cdot D \cdot S > 4m + 3 \left(\frac{2}{\lambda} - 3m \right) = \frac{6}{\lambda} - 5m \\ &> \frac{6}{\lambda} - 5 \left(4 - \frac{3}{\lambda} \right) = \frac{21}{\lambda} - 20 > 4. \end{aligned}$$

CASE B. The curve Z is reducible and contains no line passing through the point P .

(1) $Z = \tilde{C} + L_1$.

$$D_S = m\tilde{C} + m_1 L_1 + \Delta.$$

A contradictory inequality:

$$4 = T \cdot D \cdot S > 3m + m_1 + 2 \left(\frac{2}{\lambda} - m \right) = \frac{4}{\lambda} + m + m_1 > 4.$$

- (2) $Z = \widehat{C} + L_1$.
 $D_S = m\widehat{C} + m_1L_1 + \Delta$.
 An auxiliary inequality :

$$1 = L_1 \cdot D_S \geq 3m - 2m_1.$$

A contradictory inequality:

$$\begin{aligned} 4 = T \cdot D \cdot S &> 3m + m_1 + 2 \left(\frac{2}{\lambda} - 2m \right) = \frac{4}{\lambda} + m_1 - m \\ &\geq \frac{4}{\lambda} + m_1 - \frac{1 + 2m_1}{3} > 4. \end{aligned}$$

- (3) $Z = \widetilde{Q}_1 + \widetilde{Q}_2$.
 $D_S = m_1\widetilde{Q}_1 + m_2\widetilde{Q}_2 + \Delta$.
 A contradictory inequality:

$$4 = T \cdot D \cdot S > 2m_1 + 2m_2 + 2 \left(\frac{2}{\lambda} - m_1 - m_2 \right) = \frac{4}{\lambda} > 4.$$

- (4) $Z = \widetilde{Q}_1 + Q_2$.
 $D_S = m_1\widetilde{Q}_1 + m_2Q_2 + \Delta$.
 A contradictory inequality:

$$4 = T \cdot D \cdot S > 2m_1 + 2m_2 + 2 \left(\frac{2}{\lambda} - m_1 \right) = \frac{4}{\lambda} + 2m_2 > 4.$$

- (5) $Z = \widetilde{Q}_1 + L_1 + L_2$.
 $D_S = m\widetilde{Q}_1 + m_1L_1 + m_2L_2 + \Delta$.
 A contradictory inequality:

$$4 = T \cdot D \cdot S > 2m + m_1 + m_2 + 2 \left(\frac{2}{\lambda} - m \right) = \frac{4}{\lambda} + m_1 + m_2 > 4.$$

CASE C. The curve Z contains a unique line passing through the point P .

- (1) $Z = C + \widetilde{L}_1$.
 $D_S = mC + m_1\widetilde{L}_1 + \Delta$.
 An auxiliary inequality:

$$3 = C \cdot D_S \geq 3m_1.$$

A contradictory inequality:

$$\begin{aligned} 4 = T \cdot D \cdot S &> 3m + m_1 + 2 \left(\frac{2}{\lambda} - m_1 \right) = \frac{4}{\lambda} + 3m - m_1 \\ &\geq \frac{4}{\lambda} + 3m - 1 > 4. \end{aligned}$$

- (2) $Z = Q_1 + \widetilde{L}_1 + L_2$.
 $D_S = mQ_1 + m_1\widetilde{L}_1 + m_2L_2 + \Delta$.
 Auxiliary inequalities:

$$\left. \begin{aligned} 2 = Q_1 \cdot D_S &\geq -2m + 2m_1 + 2m_2 \\ 1 = L_2 \cdot D_S &\geq 2m + m_1 - 2m_2 \end{aligned} \right\} \Rightarrow 1 + m - m_2 \geq m_1$$

A contradictory inequality:

$$\begin{aligned} 4 = T \cdot D \cdot S &> 2m + m_1 + m_2 + 2 \left(\frac{2}{\lambda} - m_1 \right) = \frac{4}{\lambda} + 2m + m_2 - m_1 \\ &\geq \frac{4}{\lambda} + m + 2m_2 - 1 > 4. \end{aligned}$$

(3) $Z = \widetilde{L}_1 + L_2 + L_3 + L_4.$

$D_S = m_1 \widetilde{L}_1 + m_2 L_2 + m_3 L_3 + m_4 L_4 + \Delta$, where we may assume that $m_3 \geq m_4$.
An auxiliary inequality:

$$1 = L_4 \cdot D_S \geq m_1 + m_2 + m_3 - 2m_4.$$

A contradictory inequality:

$$\begin{aligned} 4 = T \cdot D \cdot S &> m_1 + m_2 + m_3 + m_4 + 2 \left(\frac{2}{\lambda} - m_1 \right) \\ &= \frac{4}{\lambda} + m_2 + m_3 + m_4 - m_1 \\ &\geq \frac{4}{\lambda} + 2m_2 + 2m_3 - m_4 > 4. \end{aligned}$$

(4) $Z = \widetilde{Q}_1 + \widetilde{L}_1 + L_2.$

$$D_S = m \widetilde{Q}_1 + m_1 \widetilde{L}_1 + m_2 L_2 + \Delta.$$

Auxiliary inequalities:

$$\left. \begin{aligned} 2 = Q_1 \cdot D_S &\geq -2m + 2m_1 + 2m_2 \\ 1 = L_2 \cdot D_S &\geq 2m + m_1 - 2m_2 \end{aligned} \right\} \Rightarrow 1 \geq m_1$$

A contradictory inequality:

$$\begin{aligned} 4 = T \cdot D \cdot S &> 2m + m_1 + m_2 + 2 \left(\frac{2}{\lambda} - m - m_1 \right) \\ &= \frac{4}{\lambda} + m_2 - m_1 \geq \frac{4}{\lambda} + m_2 - 1 > 4. \end{aligned}$$

(5) $Z = \widetilde{C} + \widetilde{L}_1.$

$$D_S = m \widetilde{C} + m_1 \widetilde{L}_1 + \Delta.$$

An auxiliary inequality:

$$3 = \widetilde{C} \cdot D_S \geq 3m_1.$$

A contradictory inequality:

$$\begin{aligned} 4 = T \cdot D \cdot S &> 3m + m_1 + 2 \left(\frac{2}{\lambda} - m - m_1 \right) \\ &= \frac{4}{\lambda} + m - m_1 \geq \frac{4}{\lambda} + m - 1 > 4. \end{aligned}$$

(6) $Z = \widehat{C} + \widetilde{L}_1.$

$$D_S = m \widehat{C} + m_1 \widetilde{L}_1 + \Delta.$$

Auxiliary inequalities:

$$\begin{aligned} 3 = \widehat{C} \cdot D_S &\geq 3m_1 \\ 16 &> \frac{12}{\lambda} + 3m \quad \text{by Lemma 2.3.7} \end{aligned}$$

A contradictory inequality:

$$\begin{aligned}
4 = T \cdot D \cdot S &> 3m + m_1 + 2 \left(\frac{2}{\lambda} - 2m - m_1 \right) = \frac{4}{\lambda} - m - m_1 \\
&\geq \frac{4}{\lambda} - 1 - m > \frac{4}{\lambda} - 1 - \frac{1}{3} \cdot \left(16 - \frac{12}{\lambda} \right) \\
&= \frac{8}{\lambda} - \frac{19}{3} > 4.
\end{aligned}$$

CASE D. The curve Z contains two lines passing through the point P .

(1) $Z = \widetilde{Q}_1 + \widetilde{L}_1 + \widetilde{L}_2.$

$$D_S = m\widetilde{Q}_1 + m_1\widetilde{L}_1 + m_2\widetilde{L}_2,$$

Auxiliary inequalities:

$$\begin{aligned}
2 = \widetilde{Q}_1 \cdot D_S &\geq -2m + 2m_1 + 2m_2 \\
16 &> \frac{12}{\lambda} + 2m \quad \text{by Lemma 2.3.7.}
\end{aligned}$$

A contradictory inequality:

$$\begin{aligned}
4 = T \cdot D \cdot S &> 2m + m_1 + m_2 + 2 \left(\frac{2}{\lambda} - m_1 - m_2 - m \right) \\
&= \frac{4}{\lambda} - m_1 - m_2 \geq \frac{4}{\lambda} - (1 + m) \\
&> \frac{4}{\lambda} - 1 - \left(8 - \frac{6}{\lambda} \right) = \frac{10}{\lambda} - 9 > 4.
\end{aligned}$$

(2) $Z = \widetilde{L}_1 + \widetilde{L}_2 + \widetilde{L}_3 + L_4.$

$$D_S = m_1\widetilde{L}_1 + m_2\widetilde{L}_2 + m_3\widetilde{L}_3 + m_4L_4 + \Delta, \text{ where we may assume that } m_3 \geq m_4.$$

An auxiliary inequality:

$$1 = L_4 \cdot D_S \geq m_1 + m_2 + m_3 - 2m_4$$

A contradictory inequality:

$$\begin{aligned}
4 = T \cdot D \cdot S &> m_1 + m_2 + m_3 + m_4 + 2 \left(\frac{2}{\lambda} - m_1 - m_2 \right) \\
&= \frac{4}{\lambda} + m_3 + m_4 - (m_1 + m_2) \\
&\geq \frac{4}{\lambda} + m_3 + m_4 - m_2 - (1 - m_2 - m_3 + m_4) \\
&\geq \frac{4}{\lambda} + 2m_3 - m_4 - 1 > 4.
\end{aligned}$$

(3) $Z = Q_1 + \widetilde{L}_1 + \widetilde{L}_2.$

$$D_S = mQ_1 + m_1\widetilde{L}_1 + m_2\widetilde{L}_2 + \Delta.$$

An auxiliary inequality:

$$2 = Q_1 \cdot D_S \geq -2m + 2m_1 + 2m_2 \Rightarrow 1 + m \geq m_1 + m_2.$$

A contradictory inequality:

$$\begin{aligned}
4 = T \cdot D \cdot S &> 2m + m_1 + m_2 + 2 \left(\frac{2}{\lambda} - m_1 - m_2 \right) \\
&= \frac{4}{\lambda} + 2m - m_1 - m_2 \geq \frac{4}{\lambda} + m - 1 > 4.
\end{aligned}$$

CASE E. The curve Z contains three lines passing through the point P .

- (1) $Z = \widetilde{L}_1 + \widetilde{L}_2 + \widetilde{L}_3 + L_4$.
 $D_S = m_1 \widetilde{L}_1 + m_2 \widetilde{L}_2 + m_3 \widetilde{L}_3 + m L_4$.
 Auxiliary inequalities:

$$1 = L_4 \cdot D_S \geq -2m + m_1 + m_2 + m_3$$

$$16 > \frac{12}{\lambda} + m \quad \text{by Lemma 2.3.7.}$$

A contradictory inequality:

$$\begin{aligned} 4 = T \cdot D \cdot S &> m_1 + m_2 + m_3 + m + 2 \left(\frac{2}{\lambda} - m_1 - m_2 - m_3 \right) \\ &= \frac{4}{\lambda} + m - (m_1 + m_2 + m_3) \geq \frac{4}{\lambda} + m - (1 + 2m) \\ &> \frac{4}{\lambda} - 1 - \left(16 - \frac{12}{\lambda} \right) = \frac{16}{\lambda} - 17 = 4. \end{aligned}$$

Therefore, Theorem 1.1.6 for $n = 4$ has been proved.

2.4. General quintic. In this section, we prove Theorem 1.1.6 for $n = 5$.

Let X be a quintic hypersurface in \mathbb{P}^5 such that the following general conditions hold:

- X does not contain quadric surfaces;
- X is 4-regular in the sense of [29].

Put $\lambda = \frac{22}{25}$. Let D be an effective \mathbb{Q} -divisor on X such that $D \equiv -K_X$. We claim that the log pair $(X, \lambda D)$ is log canonical.

Suppose that the log pair $(X, \lambda D)$ is not log canonical. As in the case of quartic threefolds, we may assume that $D = \frac{1}{n}R$ where R is an irreducible divisor with $R \sim -nK_X$ for some natural number n . Furthermore, the following lemma enables us to assume $n > 1$.

Lemma 2.4.1. *If a quintic hypersurface Y in \mathbb{P}^5 is 4-regular, then $\text{lct}_1(Y) = 1$.*

Proof. Let D be the hyperplane section of Y by a hyperplane $H \subset \mathbb{P}^5$. Note that D has only isolated singularities ([17]). We have to show $c(Y, D) = 1$. Because $c(Y, D) = c(H, D)$, we may consider the log pair (\mathbb{P}^4, D) instead of the log pair (Y, D) . Suppose that D has a singular point P at $[0 : 0 : 0 : 0 : 1]$. Then, it is enough to show $c_P(\mathbb{P}^4, D) = 1$. Because $c_P(\mathbb{P}^4, D)$ can be obtained in a neighborhood of P , we may consider only the log pair (\mathbb{C}^4, D') , where D' is the fifth affine piece of D . Suppose that D' is defined by a quintic polynomial $f(x, y, z, w) \in \mathbb{C}[x, y, z, w]$ that is singular at the origin. We may write

$$f(x, y, z, w) = f_2(x, y, z, w) + f_3(x, y, z, w) + f_4(x, y, z, w) + f_5(x, y, z, w),$$

where f_i is a homogeneous polynomial of degree i . Note that the 4-regularity of Y implies $f_i \neq 0$. Also, if f_2 is reduced, then we easily get $c_0(f) = 1$. Therefore, we may assume that $f_2 = x^2$. Write $f_3(x, y, z, w) = x(g_0x^2 + g_1(y, z, w)x + g_2(y, z, w)) + g_3(y, z, w)$, where g_i is a homogeneous polynomial of degree i in variables y, z , and w . Because Y is 4-regular, we can see $g_3 \neq 0$. Apply weight $w = (3, 2, 2, 2)$ to f , then we get $f_w = x^2 + g_3$. Because $c_0(f) \geq c_0(f_w) = \min\{1, c_0(x^2) + c_0(g_3)\}$, we obtain $c_0(f) = 1$ unless $g_3(y, z, w) = 0$ defines a triple line in \mathbb{P}^2 . Consequently, we may assume that

$$f = x^2 + y^3 + x(g_0x^2 + g_1x + g_2) + xh_3 + h_4 + xs_4 + s_5 + x^2(\varphi_2 + \varphi_3),$$

where h_i and s_i are homogeneous polynomials of degree i in variables y, z, w , and φ_i is a homogeneous polynomial of degree i in x, y, z, w . By applying appropriate coordinate transforms to f as in the proof of Theorem 1.1.5, we get

$$h(x, y, z, w) = x^2 + y^3 + \psi_4(y, z, w) + \psi_5(y, z, w) + \psi_6(y, z, w) + \rho_{\geq 7}(y, z, w) + x\varphi_{\geq N}(x, y, z, w)$$

that is holomorphically equivalent to f , where

$$\begin{aligned}\psi_4(y, z, w) &= h_4(y, z, w) - \frac{1}{4}g_2(y, z, w)^2, \\ \psi_5(y, z, w) &= \frac{1}{4}g_1(y, z, w)g_2(y, z, w)^2 - \frac{1}{2}g_2(y, z, w)h_3(y, z, w) - s_5(y, z, w), \\ \psi_6(y, z, w) &= g_2^2(y, z, w)\zeta_2(y, z, w) + \frac{1}{2}g_1(y, z, w)g_2(y, z, w)h_3(y, z, w) - \frac{1}{4}h_3(y, z, w)^2 \\ &\quad - \frac{1}{2}g_2(y, z, w)s_4(y, z, w)\end{aligned}$$

for some homogeneous polynomial ζ_2 of degree 2 in y, z and some polynomial $\varphi_{\geq N}$ with the lowest degree N in variables x, y, z . By applying more transforms, we may assume that N is large enough.

From now on, for any homogenous polynomial $q_i(y, z, w)$, we write

$$q_i(y, z, w) = q'_i(z, w) + y\hat{q}_i(z, w) + y^2\tilde{q}_i(y, z, w).$$

Suppose that $\psi'_4(z, w), \psi'_5(z, w), \psi'_6(z, w) = 0$, and $\hat{\psi}_4(z, w), \hat{\psi}_5(z, w) = 0$.

Because $\psi'_4(z, w) = 0$ and $\hat{\psi}_4(z, w) = 0$, we obtain

$$h_4 = y^2\tilde{h}_4 + \frac{1}{2}yg'_2 + \frac{1}{4}g_2'^2$$

from $\psi_4(y, z, w)$. Put $\check{g}_2(y, z, w) = \hat{g}_2(z, w) + y\check{g}_2(y, z, w)$. From $\psi_5(y, z, w)$, we also obtain

$$s_5 = \frac{1}{4}y \left(g'_2(g'_2\hat{g}_1 + 2g'_1\check{g}_2) - 2(h'_3\check{g}_2 + g'_2\hat{h}_3) \right) + \frac{1}{4}(g'_1g_2'^2 - 2g'_2h'_3) \quad (\text{modulo } y^2)$$

because $\psi'_5(z, w) = 0$ and $\hat{\psi}_5(z, w) = 0$. The assumption $\psi'_6(z, w) = 0$ implies that

$$4g_2'^2\zeta'_2 + 2g'_1g'_2h'_3 - h_3'^2 = 2g_2's'_4,$$

which means that h'_3 divides g'_2 , that is, $h'_3 = g'_2l_1$ for some linear form l_1 in variables y, z . Thus

$$s_5 = \frac{1}{4}yg'_2 \left(g'_2\hat{g}_1 + 2g'_1\check{g}_2 - 2l_1\check{g}_2 - 2\hat{h}_3 \right) + \frac{1}{4}g_2'^2(g'_1 - 2l_1) \quad (\text{modulo } y^2),$$

$$s'_4 = \frac{1}{2}g'_2(4\zeta'_2 + g'_1l_1 - l_1^2).$$

Applying these identities to f , we see that the polynomial f belongs to the ideal generated by $x^2, y^2, xy, xg'_2, yg'_2$, and $g_2'^2$. Then, $\text{Sing}(f(x, y, z, w) = 0) \supset (x = y = g'_2 = 0)$. However, this is impossible because $f = 0$ has only isolated singularities. Therefore, at least one of the polynomials $\psi'_4(z, w), \psi'_5(z, w), \psi'_6(z, w), \hat{\psi}_4(z, w), \hat{\psi}_5(z, w)$ must be nonzero.

(1) $\psi'_4(z, w) \neq 0$.

Use weight $w = (12, 8, 6, 6)$. Then, $h_w = x^2 + y^3 + \psi'_4(z, w)$, and hence $c_0(f) \geq c_0(h_w) = 1$

(2) $y\hat{\psi}_4(z, w) \neq 0, \psi'_4(z, w) = 0$

Using weight $w = (9, 6, 4, 4)$, we obtain $h_w = x^2 + y^3 + y\hat{\psi}_4(z, w)$. Then, $c_0(f) = 1$.

(3) $y\hat{\psi}_4(z, w) = \psi'_4(z, w) = 0$, and at least one of $\psi'_5(z, w), \psi'_6(z, w)$ is nonzero.

From weights $w = (15, 10, 6, 6)$ or $(3, 2, 1, 1)$, we can obtain $c_0(f) = 1$.

(4) $\psi'_4(z, w) = \psi'_5(z, w) = \psi'_6(z, w) = y\hat{\psi}_4(z, w) = 0$.

Using the weight $w = (3, 2, 1, 1)$, we obtain $c_0(f) = c_0(h_w = x^2 + y^3 + y\hat{\psi}_5(z, w)) = 1$.

□

Now, we get back to our proof of Theorem 1.1.6. It follows from the proof of Lemma 2.3.4 that there is a point $P \in X$ such that the log pair $(X, \lambda D)$ is log canonical on $X \setminus P$. Therefore, the log pair $(X, \lambda D)$ is not log canonical only at the point P .

The fourfold X can be given by an equation

$$w^4x + \sum_{i=2}^5 w^{5-i} q_i(x, y, z, t, u) = 0 \subset \mathbb{P}^5 \cong \text{Proj}(\mathbb{C}[x, y, z, t, u, w]),$$

where $q_i(x, y, z, t, u)$ is a homogeneous polynomial of degree i . We can assume $P = [0 : 0 : 0 : 0 : 0 : 1]$. Let T be the threefold on X given by $x = 0$.

Let $\pi: U \rightarrow X$ be the blow up at the point P and E be the π -exceptional divisor. Then

$$\bar{D} \equiv \pi^*(D) - \text{mult}_P(D) E,$$

where \bar{D} is the proper transform of the divisor D via the morphism π . Note that $\text{mult}_P(D) > \frac{1}{\lambda}$. It follows from Proposition 3 in [30] that either $\text{mult}_P(D) > \frac{2}{\lambda}$ or there is a plane $\Omega \subset E \cong \mathbb{P}^3$ such that

$$\text{mult}_P(D) + \text{mult}_\Omega(\bar{D}) > \frac{2}{\lambda}.$$

In the case when $\text{mult}_P(D) > \frac{2}{\lambda}$, let \mathcal{L} be a sufficiently general pencil of hyperplane sections of X that pass through the point P .

In the case when $\text{mult}_P(D) \leq \frac{2}{\lambda}$, let \mathcal{L} be the pencil of hyperplane sections of X such that

$$S \in \mathcal{L} \iff \text{either } \Omega \subset \bar{S} \text{ or } S = T,$$

where \bar{S} is the proper transform of S via the birational morphism π . There is a three-dimensional linear subspace $\Pi \subset \mathbb{P}^5$ such that the base locus of \mathcal{L} consists of the intersection $\Pi \cap X$.

Let S be a general threefold in \mathcal{L} . Then $S \neq T$ and $\text{mult}_P(S \cdot D) > \frac{2}{\lambda}$.

Put $Z = X|_\Pi$. The surface Z is reduced and irreducible because X contains neither quadric surfaces nor planes by our initial assumption. The 4-regularity of X implies that $\text{mult}_P(Z) \leq 3$.

Lemma 2.4.2. *The multiplicity of Z at the point P is 3.*

Proof. Suppose that $\text{mult}_P(Z) \leq 2$. Put $D \cdot S = mZ + \Upsilon$, where m is a non-negative rational number and Υ is a 2-cycle whose support does not contain the surface Z . Then

$$\text{mult}_P(\Upsilon) > \frac{2}{\lambda} - 2m,$$

but T does not contain components of Υ . Hence, we have $\text{mult}_P(T \cdot \Upsilon) > \frac{4}{\lambda} - 4m$.

Let \mathcal{M} be the linear subsystem of the linear system $|-4K_X|$ such that

$$M \in \mathcal{M} \iff \text{mult}_P(M) \geq 5$$

and let M be a general threefold in \mathcal{M} . Then the linear system \mathcal{M} contains surfaces cut out by

$$\begin{aligned} & xp_3(x, y, z, t, u) + (wx + q_2(x, y, z, t, u)) p_2(x, y, z, t, u) + \\ & + (w^2x + wq_2(x, y, z, t, u) + q_3(x, y, z, t, u)) p_1(x, y, z, t, u) + \\ & + \mu(w^3x + w^2q_2(x, y, z, t, u) + wq_3(x, y, z, t, u) + q_4(x, y, z, t, u)) = 0, \end{aligned}$$

where $p_i(x, y, z, t, u)$ is a homogeneous polynomial of degree i and $\mu \in \mathbb{C}$. Hence, the base locus of \mathcal{M} consists of finitely many lines contained in X that pass through the point P . In particular, the threefold M does not contain components of the two-cycle $T \cdot D$. Hence, it does not contain components of the one-cycle $T \cdot \Upsilon$ either. Thus, we have

$$20 - 20m = M \cdot (T \cdot D \cdot S - mT \cdot Z) = M \cdot T \cdot \Upsilon \geq \text{mult}_P(M) \text{mult}_P(T \cdot \Upsilon) > \frac{20}{\lambda} - 20m,$$

which implies an absurd inequality $\frac{22}{25} = \lambda > 1$. \square

Let \bar{T} be the proper transform of T via the birational morphism π . Then

$$3 = \text{mult}_P(Z) = \text{mult}_P(T \cdot S) = \text{mult}_P(T) \text{mult}_P(S) + \text{mult}_\Omega(\bar{T} \cdot \bar{S}),$$

which implies that $\Omega \subset \bar{T}$. Since $\text{mult}_P(D) > \frac{1}{\lambda}$ and $\text{mult}_P(T) = 2$, it follows that

$$\text{mult}_P(T \cdot D) \geq \text{mult}_P(T) \text{mult}_P(D) + \text{mult}_\Omega(\bar{T} \cdot \bar{D}) \geq 2\text{mult}_P(D) + \text{mult}_\Omega(\bar{D}) > \frac{3}{\lambda}.$$

Now we restrict everything to a general hyperplane section of the fourfold X . For the convenience, we assume that X is a smooth quintic threefold in \mathbb{P}^4 given by the equation

$$w^4x + \sum_{i=2}^5 w^{5-i} q_i(x, y, z, t) = 0 \subset \mathbb{P}^4 \cong \text{Proj}(\mathbb{C}[x, y, z, t, w]),$$

where the point P is given by the equations $x = y = z = t = 0$, and $q_i(x, y, z, t)$ is a homogeneous polynomial of degree i such that the polynomials

$$x, q_2(x, y, z, t), q_3(x, y, z, t), q_4(x, y, z, t)$$

form a regular sequence. Now T and S are hyperplane sections of X such that

$$\text{mult}_P(T) = 2, \text{mult}_P(T \cdot D) > \frac{3}{\lambda}, \text{mult}_P(S \cdot D) > \frac{2}{\lambda},$$

where D is a divisor on X such that $D \equiv \mathcal{O}_{\mathbb{P}^4}(1)|_X$.

The intersection $T \cap S$ consists of an irreducible reduced curve Z such that $\text{mult}_P(Z) = 3$. Put

$$T \cdot D = \bar{m}Z + \Upsilon,$$

where \bar{m} is a non-negative rational number and Υ is an effective one-cycle on X whose support does not contain the curve Z . Then, $\text{mult}_P(\Upsilon) > \frac{3}{\lambda} - 3\bar{m}$.

Let \mathcal{M} be the linear subsystem of the linear system $|-3K_X|$ such that

$$M \in \mathcal{M} \iff \text{mult}_P(M) \geq 4,$$

and let M be a general surface in \mathcal{M} . Then the linear system \mathcal{M} contains surfaces cut by

$$xp_2(x, y, z, t) + (wx + q_2(x, y, z, t))p_1(x, y, z, t) + \mu(w^2x + wq_2(x, y, z, t) + q_3(x, y, z, t)) = 0,$$

where $p_i(x, y, z, t)$ is a homogeneous polynomial of degree i , and $\mu \in \mathbb{C}$. In particular, the base locus of \mathcal{M} does not contain any curves because the threefold X consists no lines passing through the point P . Hence, we have

$$15 = M \cdot T \cdot D \geq 15\bar{m} + M \cdot \Upsilon > 15\bar{m} + 4\left(\frac{3}{\lambda} - 3\bar{m}\right),$$

which implies $15 > \frac{12}{\lambda} + 3\bar{m}$. Now we put

$$D_S = D|_S = mZ + \Delta,$$

where m is a non-negative rational number and Δ is an effective one-cycle on S whose support does not contain the curve Z . Then

$$5 - 5m = Z \cdot \Delta > \text{mult}_P(Z) \text{mult}_P(\Delta) > 3\left(\frac{2}{\lambda} - 3m\right),$$

because $\text{mult}_P(\Delta) > \frac{2}{\lambda} - 3m$. Thus, we see that $4m > \frac{6}{\lambda} - 5$.

The curve Z is reduced and S is a sufficiently general hyperplane section of X that contains the curve Z . Thus, we have

$$m = \text{mult}_Z(D) \leq \text{mult}_Z(D \cdot T) = \bar{m},$$

which implies

$$15 \geq \frac{12}{\lambda} + 3m > \frac{12}{\lambda} + \frac{3}{4}\left(\frac{6}{\lambda} - 5\right).$$

It contradicts $\lambda = \frac{22}{25}$.

The obtained contradiction completes the proof of Theorem 1.1.6.

3. LOG CANONICAL THRESHOLD ON A DOUBLE SPACE

3.1. Generalized global log canonical threshold. The Picard group of a Fano hypersurface of degree $n + 1 \geq 4$ of \mathbb{P}^{n+1} is generated by an anticanonical divisor. Therefore, it is natural that we consider only anticanonical divisors when we define its global log canonical threshold. However, in other varieties, it may not be enough. Therefore, we generalized the global log canonical threshold as follows:

Definition 3.1.1. Let X be a \mathbb{Q} -factorial variety with at worst log canonical singularities. For an integral divisor D on the variety X and a natural number $m > 0$, we define the m -th global log canonical threshold of the divisor D by the number⁴

$$\text{lct}_m(X, D) = \inf \left\{ c \left(X, \frac{1}{m}H \right) \mid H \in |mD| \right\}.$$

Also, we define the global log canonical threshold of D by the number

$$\text{lct}(X, D) = \inf_{n \in \mathbb{N}} \left\{ \text{lct}_n(X, D) \right\}.$$

Let $\pi : V \rightarrow \mathbb{P}^n$ be a smooth double cover ramified along a hypersurface S of degree $2m$ in \mathbb{P}^n . In addition, let H be the pull-back of a hyperplane in \mathbb{P}^n via the covering map π . One can consider the double cover V as a smooth hypersurface of degree $2m$ in $\mathbb{P}(1^{n+1}, m)$. In what follows, we will study the (first) global log canonical threshold of H on V .

3.2. Double cover of \mathbb{P}^3 ramified along a smooth sextic. Let V be the double cover of \mathbb{P}^3 ramified along a smooth sextic $S \subset \mathbb{P}^3$. Note that the pull-back of a hyperplane in \mathbb{P}^3 is an anticanonical divisor. As we did for quartic threefolds, we are to find all the possible first global log canonical thresholds of V .

For a hyperplane H in \mathbb{P}^3 , we see that

$$c(V, \pi^*(H)) = \min \left\{ 1, \frac{1}{2} + c(H, H \cap S) \right\}.$$

The intersection $H \cap S$ is a reduced sextic plane curve on $H \cong \mathbb{P}^2$. Therefore, for the first statement of Theorem 1.1.9, it is enough to consider all the possible values of $c(\mathbb{P}^2, C)$ for reduced sextic plane curves. Furthermore, we can consider only the values for $c_0(f) = c_0(\mathbb{C}^2, (f = 0))$, where f is a reduced sextic polynomial vanishing at the origin.

We may write $f(x, y) = \sum_{i=d}^6 f_i(x, y)$, where $f_i(x, y)$ is a homogenous polynomial of degree i and $f_d(x, y) \neq 0$. If $d \leq 2$, then $c_0(f) \geq \frac{1}{2}$. Also, if $d = 6$, then $c_0(f) = \frac{1}{3}$. Therefore, it is enough to consider only the case $3 \leq d \leq 5$. We see that $c_0(f) \geq \frac{1}{2}$ if $f_d(x, y) = 0$ has no root of multiplicity ≥ 3 . Consequently, by a suitable coordinate change, we may assume that $f_d(x, y) = x^3, x^4, x^3y, x^5, x^4y$, or $x^3y(ax + by)$, where $a, b \in \mathbb{C}$ with $b \neq 0$.

Because the polynomial f is reduced, it must contain a monomial of the forms either y^s or xy^t , $4 \leq s \leq 6$, $3 \leq t \leq 5$. Among such polynomials, we denote the polynomials with the lowest degrees by y^n and xy^m .

(1) $f_d(x, y) = x^3$.

(a) f contains y^n and no terms xy^t with $1 + t < n$.

Use weight $w = (n, 3)$ to get $f_w = x^3 + y^n$ or $x^3 + y^n + x^2y^2$. Because $f_w = 0$ has an isolated singularity at the origin and $n \leq 6$, $c_0(f) = \frac{n+3}{3n} \geq \frac{1}{2}$.

(b) f contains xy^m with $m \leq 4$ and no terms y^s with $s \leq m + 1$.

Use weight $w = (m, 2)$ to get $f_w = x^3 + xy^m$ or $x^3 + xy^m + x^2y^2$. Because $f_w = 0$ has an isolated singularity at the origin, $c_0(f) = \frac{m+2}{2m} \geq \frac{1}{2}$.

⁴The number $\text{lct}_m(X, D)$ is undefined if the linear system $| -mD |$ is empty.

- (c) f contains xy^m with $m = 5$ and no terms y^s with $s \leq 6$.
 If the monomial x^2y^2 belongs to f , then use weight $w = (2, 1)$ to get $f_w = x^3 + x^2y^2$.
 Because the log pair $(\mathbb{C}^2, \frac{1}{2}(f_w = 0))$ is log canonical in the outside of the origin, we obtain $c_0(f) = \frac{1}{2}$.
 If f does not contain x^2y^2 , use weight $w = (5, 2)$. Then, $f_w = x^3 + xy^5$ and $c_0(f) = \frac{7}{15}$.
- (2) $f_d(x, y) = x^4$.
- (a) f contains y^n and no terms xy^t with $1 + t < n$.
 Use weight $w = (n, 4)$ to get $f_w = x^4 + y^n$ or $x^4 + y^n + x^2y^3$. Because $f_w = 0$ has an isolated singularity at the origin, $c_0(f) = \frac{n+4}{4n}$. Therefore, the possible values are $\frac{9}{20}$ and $\frac{5}{12}$ because $n = 5$ or 6 .
- (b) f contains xy^m with $m = 4$ but not y^5 .
 Use weight $w = (4, 3)$ to get $f_w = x^4 + xy^4$ or $x^4 + xy^4 + x^2y^3$. Because $f_w = 0$ has an isolated singularity at the origin, $c_0(f) = \frac{7}{16}$.
- (c) f contains xy^m with $m = 5$ and no terms y^s with $s \leq 6$.
 If the monomial x^2y^3 belongs to f , then use weight $w = (3, 2)$ to get $f_w = x^3 + x^2y^3$.
 Because the log pair $(\mathbb{C}^2, \frac{5}{12}(f_w = 0))$ is log canonical in the outside of the origin, we obtain $c_0(f) = \frac{5}{12}$.
 If f does not contain x^2y^3 , then use weight $w = (5, 3)$. Then $f_w = x^4 + xy^5$ and $c_0(f) = \frac{2}{5}$.
- (3) $f_d(x, y) = x^3y$.
- (a) f contains y^n and no terms xy^t with $1 + t < n$.
 Use weight $w = (n - 1, 3)$ to get $f_w = x^3y + y^n$. Because $f_w = 0$ has an isolated singularity at the origin, $c_0(f) = \frac{n+2}{3n}$. Therefore, the possible values are $\frac{7}{15}$ and $\frac{4}{9}$ because $n = 5$ or 6 .
- (b) f contains xy^m with $m = 4$ but not y^5 .
 Use weight $w = (3, 2)$ to get $f_w = x^3y + xy^4$. Because $f_w = 0$ has an isolated singularity at the origin, $c_0(f) = \frac{5}{11}$.
- (c) f contains xy^m with $m = 5$ and no terms y^s with $s \leq 6$.
 Use weight $w = (2, 1)$. Then, $f_w = x^3y + xy^5$ or $x^3y + xy^5 + x^2y^3$. Because the log pair $(\mathbb{C}^2, \frac{3}{7}(f_w = 0))$ is log canonical in the outside of the origin, we obtain $c_0(f) = \frac{3}{7}$.
- (4) $f_d(x, y) = x^5$.
- (a) f contains y^6 .
 Use weight $w = (6, 5)$. Then, we see $c_0(f) = \frac{11}{30}$.
- (b) f contains xy^5 but not y^6 .
 Use weight $w = (5, 4)$. Then, we see $c_0(f) = \frac{9}{25}$.
- (5) $f_d(x, y) = x^4y$.
- (a) f contains y^6 .
 Use weight $w = (5, 4)$. Then, we see $c_0(f) = \frac{3}{8}$.
- (b) f contains xy^5 but not y^6 .
 Use weight $w = (4, 3)$. Then, we see $c_0(f) = \frac{7}{19}$.
- (6) $f_d(x, y) = x^3y(ax + by)$.
- (a) f contains y^6 .
 Use weight $w = (4, 3)$ to get $f_w = x^3y^2 + y^6$. Because the log pair $(\mathbb{C}^2, \frac{7}{18}(f_w = 0))$ is log canonical in the outside of the origin, we obtain $c_0(f) = \frac{7}{18}$.
- (b) f contains xy^5 but not y^6 .
 Use weight $w = (3, 2)$ to get $f_w = x^3y^2 + xy^5$. Because the log pair $(\mathbb{C}^2, \frac{5}{13}(f_w = 0))$ is log canonical in the outside of the origin, we obtain $c_0(f) = \frac{5}{13}$.

We have seen all the possible log canonical thresholds $c(H, H \cap S)$. The table below shows that for each number μ of the set in Theorem 1.1.9 there exists a reduced sextic curve C in \mathbb{P}^2 with $c(\mathbb{P}^2, C) = \mu - \frac{1}{2}$.

μ	Equation	μ	Equation
$\frac{5}{6}$	$x^6 + y^6 + z^6$	$\frac{43}{50}$	$x^5z + xy^5 + x^6$
$\frac{13}{15}$	$x^5z + y^6 + x^6$	$\frac{33}{38}$	$x^4yz + xy^5 + x^6$
$\frac{7}{8}$	$x^4yz + y^6 + x^6$	$\frac{8}{9}$	$x^3y^2z + y^6 + x^4yz$
$\frac{23}{26}$	$x^3y^2z + xy^5 + x^4yz$	$\frac{9}{10}$	$x^4z^2 + y^5z + x^6$
$\frac{11}{12}$	$x^4z^2 + y^6 + x^6$	$\frac{15}{16}$	$x^4z^2 + xy^4z + x^6 + y^6$
$\frac{19}{20}$	$x^4z^2 + y^5z + x^6$	$\frac{13}{14}$	$x^3yz^2 + xy^5$
$\frac{17}{18}$	$x^3yz^2 + y^6$	$\frac{21}{22}$	$x^3yz^2 + xy^4z + x^6 + y^6$
$\frac{29}{30}$	$x^3yz^2 + y^5z + x^6$		

For the sextic curve C defined by each equation in the table above, one can easily construct a smooth sextic surface $S \subset \mathbb{P}^3$ that has a hyperplane H in \mathbb{P}^3 with $H \cap S = C$. Then, for the double cover $\pi : V \rightarrow \mathbb{P}^3$ ramified along the sextic S , the log canonical threshold $c(V, \pi^*(H))$ is exactly μ . Moreover, using the same method for Lemma 2.2.2, one can prove the second statement of Theorem 1.1.9.

Lemma 3.2.1. *Let $f_6(x, y, z)$ be a reduced sextic homogenous polynomial. And let V be the double cover of \mathbb{P}^3 ramified along the sextic surface defined by $f_6(x, y, z) + wg_5(x, y, z, w) = 0$, where g_5 is a general quintic homogenous polynomial. Then we have $\text{lct}_1(V) = c(V, D)$, where the divisor $D \subset V$ is the pull-back of the hyperplane in \mathbb{P}^3 given by $w = 0$.*

Proof. As in the proof of Proposition 2.2.2, let

$$\mathcal{S} = \left\{ ((a_1, a_2, a_3), G) \in \mathbb{C}^3 \times H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(5)) \mid \begin{array}{l} f_6(a_1, a_2, a_3) + G(a_1, a_2, a_3, 1) = 0 \\ f_6 + wG = 0 \text{ is smooth.} \end{array} \right\}$$

with the natural projections $p : \mathcal{S} \rightarrow H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(5))$ and $q : \mathcal{S} \rightarrow \mathbb{C}^3$. Let $P = [a_1 : a_2 : a_3 : a_4]$ be a point on the sextic surface defined by $F = f_6 + wG$ with $a_4 \neq 0$. Then, we may assume that the point P is located at $[0 : 0 : 0 : 1]$ and that the sextic homogeneous polynomial F can be given by

$$w^5x + w^4q_2(x, y, z) + w^3q_3(x, y, z) + w^2q_4(x, y, z) + wq_5(x, y, z) + f_6(x, y, z),$$

where each g_i is a homogeneous polynomial of degree i . We will say that the point P is *bad* on the sextic surface $F = 0$ if the following condition holds:

- the quadratic form $q_2(0, y, z)$ is zero and $q_3(0, y, z) = l(y, z)^3$, where l is a linear form.

Put

$$\mathcal{Y} = \{((a_1, a_2, a_3), G) \in \mathcal{S} \mid [a_1 : a_2 : a_3 : 1] \text{ is bad on the sextic } f_6 + wG = 0\}.$$

Since the condition above is 5-dimensional, the restriction

$$q|_{\mathcal{Y}} : \mathcal{Y} \rightarrow H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(5))$$

is not surjective. Therefore, for a general quintic homogenous polynomial G , the sextic surface S defined by $f_6 + wG = 0$ has no bad point. Then, one can easily check that the first global log canonical threshold of the double cover V of \mathbb{P}^3 ramified along the sextic S is equal to the log canonical threshold of the pull-back of the hyperplane defined by $w = 0$ in \mathbb{P}^3 on V . \square

3.3. Double spaces. Let $\pi : V \rightarrow \mathbb{P}^n$ be a smooth double cover ramified along a hypersurface S of degree $2m$ in \mathbb{P}^n , $n \geq 2$. In addition, let H be the pull-back of a hyperplane in \mathbb{P}^n by the covering map π .

Proposition 3.3.1. *The global log canonical threshold $\text{lct}(V, H)$ is equal to the first global log canonical threshold $\text{lct}_1(V, H)$.*

Proof. Let us use the arguments in [30]. Suppose that there is a divisor D in the linear system $|\mu H|$ such that

$$c\left(V, \frac{1}{\mu}D\right) \leq \text{lct}_1(V, H) \leq 1$$

where $\mu \geq 2$ is a natural number. Put $\lambda = c\left(V, \frac{1}{\mu}D\right)$. Then, the log pair $\left(X, \frac{\lambda}{\mu}D\right)$ is not log terminal.

We claim that $D = \mu T$ for some divisor T in the linear system $|H|$, which implies that $c\left(V, \frac{1}{\mu}D\right) = \text{lct}_1(V, H)$.

Suppose that it is not a case. Then it follows from Remark 2.3.3 that we may assume that the support of the divisor D does not contain divisors of the linear system $|H|$. Then there is a point $P \in V$ such that one of the following holds:

- the point P is a center of log canonical singularities of $\left(V, \frac{\lambda}{\mu}D\right)$, and $\pi(P) \notin S$;
- the inequality $\text{mult}_P(D) > \mu$ holds, and $\pi(P) \in S$.

Suppose that $\pi(P) \in S$. Let T be the unique divisor in the linear system $|H|$ that is singular at the point P . Then, we obtain an absurd inequality

$$2\mu = D \cdot T^{n-1} \geq \text{mult}_P(D \cdot T) > 2\mu.$$

Now, we suppose that $\pi(P) \notin S$ and the point P is a center of log canonical singularities of the log pair $\left(V, \frac{\lambda}{\mu}D\right)$.

Let $\xi : W \rightarrow V$ be the blow up at the point P and $E \cong \mathbb{P}^{n-1}$ be the exceptional divisor of the birational morphism ξ . Then, it follows from Proposition 3 in [30] that there is a hyperplane $\Lambda \subset E$ such that

$$(3.3.2) \quad \text{mult}_P(D) + \text{mult}_\Lambda(\bar{D}) > 2\mu,$$

where \bar{D} is the proper transform of D on the variety W .

Let G be a general divisor in $|H|$ such that $\Lambda \subset \text{Supp}(\bar{G})$, where \bar{G} is the proper transform of G on the variety W . Then, we also obtain a contradictory inequality

$$2\mu = D \cdot G^{n-1} \geq \text{mult}_P(D \cdot G) > 2\mu.$$

□

Because the first global log canonical thresholds coincide with the global log canonical thresholds on double spaces, Theorem 1.1.9 implies a stronger result as follows.

Corollary 3.3.3. *Let V be a smooth double cover of \mathbb{P}^3 ramified along a sextic. Then, the global log canonical threshold of the Fano variety V is one of the following:*

$$\left\{ \frac{5}{6}, \frac{43}{50}, \frac{13}{15}, \frac{33}{38}, \frac{7}{8}, \frac{33}{38}, \frac{8}{9}, \frac{9}{10}, \frac{11}{12}, \frac{13}{14}, \frac{15}{16}, \frac{17}{18}, \frac{19}{20}, \frac{21}{22}, \frac{29}{30}, 1 \right\}.$$

Furthermore, for each number μ in the set above, there is a smooth double cover V of \mathbb{P}^3 ramified along a sextic with $\text{lct}(V) = \mu$.

Now we are ready to prove the following result.

Proposition 3.3.4. *The following inequality holds:*

$$\text{lct}(V, H) \geq \min \left(1, \frac{m+n-1}{2m} \right).$$

Proof. By Proposition 3.3.1, it is enough to consider the first global log canonical threshold $\text{lct}_1(V, H)$ instead of $\text{lct}(V, H)$. Let D be a divisor in $|H|$.

The double space V can be given by a quasi-homogenous equation

$$w^2 = f(x_0, \dots, x_n) \subset \mathbb{P}(1^{n+1}, m) \cong \text{Proj}(\mathbb{C}[x_0, \dots, x_n, w]),$$

where $\text{wt}(x_i) = 1$, $\text{wt}(w) = m$, and f is a homogeneous polynomial of degree $2m$. Then, we may assume that the divisor D is cut out on V by the equation $x_0 = 0$. The divisor D is a hypersurface

$$w^2 = f(0, x_1, \dots, x_n) \subset \mathbb{P}(1^n, m) \cong \text{Proj}(\mathbb{C}[x_1, \dots, x_n, w]),$$

which has isolated singularities because the hypersurface

$$f(0, x_1, \dots, x_n) = 0 \subset \mathbb{P}^{n-1} \cong \text{Proj}(\mathbb{C}[x_1, \dots, x_n]),$$

has isolated singularities (see [12], [17], [34], Theorem 2 in [28]).

It follows from [11] that the log pair $(V, \lambda D)$ is log terminal if and only if $(\mathbb{P}(1^n, m), \lambda D)$ is log terminal because V is smooth and the divisor D is contained in the smooth locus of $\mathbb{P}(1^n, m)$. Then

$$c(V, D) = c(\mathbb{P}(1^n, m), D) = \frac{1}{2} + \text{lct}(\mathbb{P}^{n-1}, f(0, x_1, \dots, x_n) = 0) \geq \frac{1}{2} + \frac{n-1}{2m}$$

by [2] and Proposition 8.21 in [19]. □

Let $\pi: V \rightarrow \mathbb{P}^n$ be a double cover ramified along a smooth hypersurface of degree $2n \geq 4$. It is a Fano variety of Fano index 1 and the pull-back of a hyperplane in \mathbb{P}^n is an anticanonical divisor of V . It follows from Proposition 3.3.4 that

$$\text{lct}(V) \geq \frac{2n-1}{2n},$$

while it follows from [30] that $\text{lct}(V) = 1$ if V is general and $n \geq 3$.

Therefore, Proposition 3.3.4 immediately implies the following result that has been proved in [1].

Corollary 3.3.5. *A smooth double cover of \mathbb{P}^n ramified along a hypersurface of degree $2n \geq 4$ admits a Kähler-Einstein metric.*

The following result follows from [2] and the proof of Proposition 3.3.4.

Corollary 3.3.6. *Let V be the smooth hypersurface in $\mathbb{P}(1^{n+1}, m)$ of degree $2m \geq 2n \geq 6$ given by an equation*

$$w^2 = f(x_0, \dots, x_n) \subset \mathbb{P}(1^{n+1}, m) \cong \text{Proj}(\mathbb{C}[x_0, \dots, x_n, w]),$$

where $\text{wt}(x_i) = 1$, $\text{wt}(w) = m$, and f is a homogeneous polynomial of degree $2m$. Suppose that

$$c(V, D) = \frac{m+n-1}{2m\mu},$$

where $D \in |\mu H|$ and $\mu \in \mathbb{N}$. Then $D = \mu T$, where T is a divisor that is cut out on the hypersurface V by an equation $\sum_{i=0}^n \lambda_i x_i = 0$ such that the hypersurface

$$f(x_0, \dots, x_n) = \sum_{i=0}^n \lambda_i x_i = 0 \subset \mathbb{P}^{n-1} \cong \text{Proj} \left(\mathbb{C}[x_0, \dots, x_n] / \left(\sum_{i=0}^n \lambda_i x_i \right) \right)$$

is a cone over a smooth hypersurface in \mathbb{P}^{n-2} of degree $2m$.

We conclude this section by proving an easy result, which follows from [30].

Proposition 3.3.7. *Suppose that V is general and $n \geq 3$. Then $\text{lct}(V, H) = 1$.*

Proof. We assume that for every hyperplane $M \subset \mathbb{P}^n$, the intersection $S \cap M$ has at most isolated double points. This generality condition is obviously satisfied for general choice of the double cover V because $n \geq 3$.

Let D be a divisor in the linear system $|H|$. It follows from Lemma 8.12 in [19] that the singularities of the log pair (V, D) are log canonical if and only if the singularities of the log pair

$$\left(\mathbb{P}^n, \pi(D) + \frac{1}{2}S \right)$$

are log canonical. Put $M = \pi(D)$. It follows from Theorem 7.5 in [19] that the singularities of the log pair (V, D) are log canonical if and only if the log pair $(M, \frac{1}{2}S|_M)$ is log canonical. But the log pair $(M, \frac{1}{2}S|_M)$ is log canonical because $S|_M$ has at most isolated double points. \square

The generality assumption in Proposition 3.3.7 is weaker than the one in [30].

Example 3.3.8. Let V be the smooth double cover of \mathbb{P}^3 ramified along the sextic surface $S \subset \mathbb{P}^3$ defined by the equation

$$x_0^6 + x_1^6 + x_2^6 + x_3^6 + x_0^2 x_1^2 x_2 x_3 = 0.$$

Let $C \subset \mathbb{P}^3$ be the curve defined by the intersection of the surface S and the Hessian surface $\text{Hess}(S)$ of S . For the tangent hyperplane T_P at a point $P \in S$, if the multiplicity of the curve $T_P \cap S$ at the point P is at least 3, then the curve C is singular at the point P . Using the computer program, *Singular*, one can check that the curve C is smooth in the outside of the curves $x_i = x_j = 0$ with $i \neq j$. Furthermore, for a point P in S that belongs to the curves $x_i = x_j = 0$ with $i \neq j$, one can easily check that the log pair $(S, \frac{1}{2}H_P)$ is log canonical, where H_P is the hyperplane section of S by the tangent hyperplane to S at the point P . Consequently, $\text{lct}(V) = \text{lct}_1(V) = 1$. The variety V is an explicit example of smooth Fano variety with the following properties (We do not know any other explicit example of such a smooth Fano variety). For each $i = 1, 2, \dots, r$, let $V_i = V$. Then, the paper [30] implies that the product $V_1 \times \dots \times V_r$ is not rational and

$$\text{Bir}(V_1 \times \dots \times V_r) = \text{Aut}(V_1 \times \dots \times V_r).$$

Moreover, for each dominant rational map $\rho : V_1 \times \dots \times V_r \dashrightarrow Y$ whose general fiber is rationally connected, there is a subset $\{i_1, \dots, i_k\} \subset \{1, \dots, r\}$ such that the diagram

$$\begin{array}{ccc} V_1 \times \dots \times V_r & & \\ \pi \downarrow & \searrow \rho & \\ V_{i_1} \times \dots \times V_{i_k} & \xrightarrow{\bar{\rho}} & Y \end{array}$$

commutes, where π is the natural projection and $\bar{\rho}$ is a birational map.

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